Inexact Newton's method to nonlinear functions with values in a cone

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Abstract

The problem of finding a solution of nonlinear inclusion problems in Banach space is considered in this paper. Using convex optimization techniques introduced by Robinson (Numer. Math., Vol. 19, 1972, pp. 341-347), a robust convergence theorem for inexact Newton's method is proved. As an application, an affine invariant version of Kantorovich's theorem and Smale's α-theorem for inexact Newton's method is obtained.

Keywords: Inclusion problems, inexact Newton's method, majorant condition, semi-local convergence.

1 Introduction

In this paper we study the inexact Newton's method for solving the nonlinear inclusion problem

$$F(x) \in C, \tag{1}$$

where $F : \Omega \to \mathbb{Y}$ is a nonlinear continuously differentiable function, $\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces, $\mathbb{X}$ is reflexive, $\Omega \subseteq \mathbb{X}$ an open set and $C \subset \mathbb{Y}$ a nonempty closed convex cone. The idea of solving a nonlinear inclusion problems of the form (1), plays a huge role in classical analysis and its applications. For instance, the special case in which C is the degenerate cone $\{0\} \subset \mathbb{Y}$, the inclusion problem in (1) correspond to a nonlinear equation. In the case for which $\mathbb{X} = \mathbb{R}^n$, $\mathbb{Y} = \mathbb{R}^{p+q}$ and $C = \mathbb{R}^p_- \times \{0\}$ is the product of the nonpositive orthant in $\mathbb{R}^p$ with the origin in $\mathbb{R}^q$, the inclusion problem in (1) correspond to a nonlinear systems of p inequalities and q equalities, for example see [2, 7, 4, 8, 9, 16, 19, 20].

In order to solving (1), Robinson in [22] proposed the following iterative methods of Newton-type:

$$x_{k+1} = x_k + d_k, \qquad d_k \in \operatorname*{arg\,min}_{d \in \mathbb{X}} \left\{ \|d\| \ : \ F(x_k) + F'(x_k)d \in C \right\}, \qquad k = 0, 1, \ldots. \tag{2}$$

In general, this algorithm may fail to converge and may even fail to be well defined. To ensure that the method is well defined and converges to a solution of a given nonlinear inclusion, Robinson made two important assumptions:

H1. There exists $x_0 \in \mathbb{X}$ such that rge $T_{x_0} = \mathbb{Y}$, where $T_{x_0} : \mathbb{X} \rightrightarrows \mathbb{Y}$ is the convex process given by

$$T_{x_0} d := F'(x_0)d - C, \qquad d \in \mathbb{X},$$

and rge $T_{x_0} = \{y \in \mathbb{Y} \ : \ y \in T_{x_0}(x)$ for some $x \in \mathbb{X}\}$, see [9] for adicional details.

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H2. F' is Lipschitz continuous with modulo L, i.e., $\|F'(x) - F'(y)\| \le L\,\|x - y\|$, for all $x, y, \in \mathbb{X}$.

Under these assumptions, it was proved in [22], that the sequence $\{x_k\}$ generate by (2) is well defined and converges to x_* satisfying $F(x_*) \in C$, provided that following convergent criterion is satisfied:

$$\|x_1 - x_0\| \le \frac{1}{2L\|T_{x_0}^{-1}\|}.$$

The first affine invariante version in this result was presented by Li and Ng in [20]. In [21] Li and Ng introduced the notion of the weak-Robinson condition for convex processes and presented a extension of the results of [20] under a L-average Lipschitz condition. As an applications, two special cases were provided, namely, the convergence result of the method under Lipschitz's condition and Smale's condition. In [12], under a affine majorant condition, a robust analysis of this method were established. As in [20], the analysis under Lipschitz's condition and Smale's condition are also obtained as special case, see also [1, 5].

The inexact Newton method, for solving nonlinear equation $F(x) = 0$, was introduced by Dembo, Eisenstat, and Steihaug in [6] for denoting any method which, given an initial point x_0, generates the sequence $\{x_k\}$ as follows:

$$\|F(x_k) + F'(x_k)(x_{k+1} - x_k)\| \le \eta_k \|F(x_k)\|, \qquad k = 0, 1, \ldots, \tag{3}$$

and $\{\eta_k\}$ is a sequence of forcing terms such that $0 \le \eta_k < 1$. In [6] was proved, under suitable assumptions, that for any starting point x_0 in a certain neighborhood of the solution x_*, the sequence $\{x_k\}$ generated by (3) is convergent for this solution with super-linear rate. In [18] numerical issues about this method are discussed.

In the present paper, we extend the inexact Newton's method (3), for solving nonlinear inclusion, as any method which, given an initial point x_0, generates a sequence $\{x_k\}$ as follows:

$$x_{k+1} = x_k + d_k, \qquad d_k \in \operatorname*{arg\,min}_{d \in \mathbb{X}} \left\{ \|d\| \ : \ F(x_k) + F'(x_k)d + r_k \in C \right\}, \tag{4}$$

$$\max_{w \in \{-r_k,\ r_k\}} \left\|T_{x_0}^{-1} w\right\| \le \theta \left\|T_{x_0}^{-1}[-F(x_k)]\right\|, \tag{5}$$

for $k = 0, 1, \ldots$, $0 \le \theta < 1$ is a fixed suitable tolerance, and

$$T_{x_0}^{-1}(y) := \left\{ d \in \mathbb{X} \ : \ F'(x_0)d - y \in C \right\}, \ y \in \mathbb{Y}.$$

The analysis of this method, under Lipschitz's condition and Smale's condition, are provided as special case. It is well known that the mapping $T_{x_0}^{-1}$ is convex process with closed graphic; see Corollary 4A.7 of [9]. Hence, if $\theta = 0$ then (4)-(5) reduces to extended Newton method (2) for solving (1), and, in the case, $C = \{0\}$ it reduces to affine invariante version of (3), which was studied by Ferreira and Svaiter in [15]. Up to our knowledge, this is the first time that the inexact Newton method to solving cone inclusion problems with a relative error tolerance is analyzed.

The problem in (1) is a particular instance of the following generalized equation

$$F(x) + C(x) \ni 0, \tag{6}$$

when $C(x) \equiv -C$, where $C : \mathbb{X} \rightrightarrows \mathbb{Y}$ is a set valued mapping. In [11] (see also [3]), Dontchev and Rockafellar proposed the following iterative methods of Newton-type for solving (6):

$$(F(x_k) + F'(x_k)(x_{k+1} - x_k) + C(x_{k+1})) \cap R_k(x_k, x_{k+1}) \neq \varnothing, \qquad k = 0, 1, \ldots, \tag{7}$$

where $R_k : \mathbb{X} \times \mathbb{X} \rightrightarrows \mathbb{Y}$ is a sequence of set-value mappings with closed graphs. Note that, in the case, when $C(x) \equiv 0$, $\theta \equiv \eta_k$ and

$$R_k(x_k, x_{k+1}) \equiv B_{\eta_k \|f(x_k)\|}(0),$$

the iteration (7) reduces to (3). In the case $C(x) \equiv -C$, the iteration (7) has (4)-(5) as a minimal norm affine invariante version. Therefore, in some sense, our method is a particular case of [11]. However, the analysis presented in [11] is local, i.e, it is made assumption at a solution, while in our analysis we will not assume existence of solution. In fact, our aim is to prove a robust Kantorovich's Theorem for (4)-(5), under assumption **H1** and an affine invariant majorant condition generalizing **H2**, which in particular, prove existence of solution for (1). Moreover, the analysis presented, show that the robust analysis of the inexact Newton's method for solving nonlinear inclusion problems, under affine Lipschitz-like and affine Smale's conditions, can be obtained as a special case of the general theory. Besides, for the degenerate cone, which the nonlinear inclusion becomes a nonlinear equation, our analysis retrieves the classical results on semi-local analysis of inexact Newton's method; [15].

The organization of the paper is as follows. In Section 1.1, some notations and basic results used in the paper are presented. In Section 2, the main results are stated and in Section 2.1 some properties of the majorant function are established and the main relationships between the majorant function and the nonlinear operator used in the paper are presented. In Section 3, the main results are proved and the applications of this results are given in Section 4. Some final remarks are made in Section 5.

1.1 Notation and auxiliary results

The following notations and results are used throughout our presentation. We beginning with the following elementary convex analysis result:

Proposition 1. *Let $I \subset \mathbb{R}$ be an interval and $\varphi : I \to \mathbb{R}$ be convex. For any $s_0 \in \operatorname{int}(I)$, the left derivative there exist (in $\mathbb{R}$)*

$$D^- \varphi(s_0) := \lim_{s \to s_0^-} \frac{\varphi(s_0) - \varphi(s)}{s_0 - s} = \sup_{s < s_0} \frac{\varphi(s_0) - \varphi(s)}{s_0 - s}.$$

Moreover, if $s, t, r \in I$, $s < r$, and $s \leqslant t \leqslant r$ then $\varphi(t) - \varphi(s) \leqslant [\varphi(r) - \varphi(s)]\,[(t - s)/(r - s)]$.

Proof. See [17]. □

Proposition 2. *Let $\{x_k\}$ be a sequence in $\mathbb{X}$ and $\Theta > 0$. If $\{x_k\}$ converges to x_* and satisfies*

$$\|x_{k+1} - x_k\| \leq \Theta \|x_k - x_{k-1}\|, \qquad k = 1, 2, \ldots, \tag{8}$$

then $\{x_k\}$ converges Q-linearally to z_* as follows

$$\limsup_{k \to \infty} \frac{\|x_{k+1} - x_*\|}{\|x_k - x_*\|} \leq \Theta.$$

Proof. The proof follows the same pattern as the proof of Proposition 2 of [13]. □

Let $\mathbb{X}$ be a Banach space. The *open* and *closed ball* at x with radius $\delta > 0$ are denoted, respectively, by

$$B(x, \delta) := \{y \in \mathbb{X} \;:\; \|x - y\| < \delta\} \quad \text{and} \quad B[x, \delta] := \{y \in \mathbb{X} \;:\; \|x - y\| \leqslant \delta\}.$$

Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces. A set valued mapping $T : \mathbb{X} \rightrightarrows \mathbb{Y}$ is called *sublinear* or *convex precess* when its graph is a convex cone, i.e.,

$$0 \in T(0), \qquad T(\lambda x) = \lambda T(x), \qquad \lambda > 0, \qquad T(x + x') \supseteq T(x) + T(x'), \quad x, x' \in \mathbb{X}, \tag{9}$$

(sublinear mapping has been extensively studied in [9], [23], [24] and [25]). The following definitions and results about sublinear mappings will be need: The *domain* and *range* of a sublinear mapping T are defined, respectively, by

$$\operatorname{dom} T := \{d \in \mathbb{X} \;:\; Td \neq \varnothing\}, \qquad \operatorname{rge} T := \{y \in \mathbb{Y} \;:\; y \in T(x) \text{ for some } x \in \mathbb{X}\}.$$

The *inverse* $T^{-1} : \mathbb{Y} \rightrightarrows \mathbb{X}$ of a sublinear mapping T is another sublinear mapping defined by

$$T^{-1}y := \{d \in \mathbb{X} \ : \ y \in Td\}, \quad y \in \mathbb{Y}.$$

The *norm* (or inner norm as is called in see [9]) of a sublinear mapping T is defined by

$$\|T\| := \sup \{\|Td\| \ : \ d \in \operatorname{dom} T, \ \|d\| \leqslant 1\}, \tag{10}$$

where $\|Td\| := \inf\{\|v\| \ : \ v \in Td\}$ for $Td \neq \varnothing$. We use the convention $\|Td\| = +\infty$ for $Td = \varnothing$, it will be also convenient to use the convention $Td + \varnothing = \varnothing$ for all $d \in \mathbb{X}$.

Lemma 3. *Let $T : \mathbb{X} \rightrightarrows \mathbb{Y}$ be a sublinear mapping with closed graph. Then $dom T = \mathbb{X}$ if and only if $\|T\| < +\infty$ and $rge T = \mathbb{Y}$ if and only if $\|T^{-1}\| < +\infty$.*

Proof. See Corollary 5C.2 of [9]. □

Let $S, T : \mathbb{X} \rightrightarrows \mathbb{Y}$ and $U : \mathbb{Y} \rightrightarrows \mathbb{Z}$ be sublinear mappings. The scalar *multiplication*, *addition* and *composition* of sublinear mappings are sublinear mappings defined, respectively, by

$$(\alpha S)(x) := \alpha S(x), \quad (S+T)(x) := S(x) + T(x), \quad UT(x) := \bigcup \{U(y) \ : \ y \in T(x)\},$$

for all $x \in \mathbb{X}$ and $\alpha > 0$ and the following norm properties there hold:

$$\|\alpha S\| = |\alpha| \|S\|, \quad \|S + T\| \leqslant \|S\| + \|T\|, \quad \|UT\| \leqslant \|U\| \|T\|.$$

Remark 1. *Note that definition of the norm in* (10) *implies that if $dom T = \mathbb{X}$ and A is a linear mapping from $\mathbb{Z}$ to $\mathbb{X}$ then $\|T(-A)\| = \|TA\|$.*

Lemma 4. *Let $S, T : \mathbb{X} \rightrightarrows \mathbb{Y}$ be a sublinear mappings with closed graph such that $dom S = dom T = X$ and $\|T^{-1}\| < +\infty$. Suppose that $\|T^{-1}\|\|S\| < 1$ and $(S+T)(x)$ is closed for each $x \in \mathbb{X}$ then $rge (S+T)^{-1} = \mathbb{X}$ and*

$$\|(S+T)^{-1}\| \leqslant \frac{\|T^{-1}\|}{1 - \|T^{-1}\|\|S\|}.$$

Proof. These results follows from Theorem 5 of [23] by taking into account Lemma 3. □

Lemma 5. *Let $G : [0,1] \to \mathbb{Y}$ and $g : [0,1] \to \mathbb{R}$ be continuous function and $\mathbb{Z}$ a reflexive Banach space. Suppose that $U : \mathbb{Y} \rightrightarrows \mathbb{Z}$ is a sublinear mapping with closed graphic such that $dom U \supseteq rge G$. If*

$$\|UG(\tau)\| \leqslant g(\tau), \qquad \tau \in [0,1],$$

then there hold

$$U \int_0^1 G(\tau) d\tau \neq \varnothing, \qquad \left\| U \int_0^1 G(\tau) d\tau \right\| \leqslant \int_0^1 g(\tau) d\tau.$$

Proof. See Lemma 2.1 of [20]. □

Let $\Omega \subseteq \mathbb{X}$ be an open set and $F : \Omega \to \mathbb{Y}$ a continuously Fréchet differentiable function. The linear map $F'(x) : \mathbb{X} \to \mathbb{Y}$ denotes the Fréchet derivative of $F : \Omega \to \mathbb{Y}$ at $x \in \Omega$. Let $C \subset \mathbb{Y}$ be a nonempty closed convex cone, $z \in \Omega$ and $T_z : \mathbb{X} \rightrightarrows \mathbb{Y}$ a mapping defined as

$$T_z d := F'(z)d - C. \tag{11}$$

It is well known that the mappings T_z and T_z^{-1} are sublinear with closed graphic, $\operatorname{dom} T_z = X$, $\|T_z\| < +\infty$ and, moreover, $\operatorname{rge} T_z = Y$ if and only if $\|T_z^{-1}\| < +\infty$ (see Lemma 3 above and Corollary 4A.7, Corollary 5C.2 and Example 5C.4 of [9]). Note that

$$T_z^{-1} y := \{d \in \mathbb{X} \ : \ F'(z)d - y \in C\}, \qquad z \in \Omega, \ y \in \mathbb{Y}. \tag{12}$$

Lemma 6. *Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces, $\Omega \subseteq \mathbb{X}$ an open set and $F : \Omega \to \mathbb{Y}$ a continuously Fréchet differentiable function. Then the following inclusion holds*

$$T_z^{-1}F'(v)T_v^{-1}w \subseteq T_z^{-1}w, \qquad v, z \in \Omega,\ w \in \mathbb{Y}.$$

As a consequence,

$$\left\|T_z^{-1}\left[F'(y) - F'(x)\right]\right\| \leq \left\|T_z^{-1}F'(v)T_v^{-1}\left[F'(y) - F'(x)\right]\right\|, \qquad v, x, y, z \in \Omega.$$

Proof. See [12]. □

2 Kantorovich's Theorem for Inexact Newton's method

Our goal is to state and prove a robust semi-local affine invariant theorem for inexact Newton's method to solve nonlinear inclusion of the form (1), for state this theorem we need some definitions.

A nonlinear continuously Fréchet differentiable function $F : \Omega \to \mathbb{Y}$ satisfies the *Robson's Condition* at $x_0 \in \Omega$ if

$$\operatorname{rge} T_{x_0} = \mathbb{Y},$$

where $T_{x_0} : \mathbb{X} \rightrightarrows \mathbb{Y}$ is a sublinear mapping as defined in (11).

Let $\mathbb{X}$ and $\mathbb{Y}$ be a Banach spaces, $\Omega \subseteq \mathbb{X}$ an open set and $R > 0$ a scalar constant. A continuously differentiable scalar function $f : [0, R) \to \mathbb{R}$ is a *majorant function* at a point $x_0 \in \Omega$ for a continuously differentiable function $F : \Omega \to \mathbb{Y}$ if

$$B(x_0, R) \subseteq \Omega, \qquad \left\|T_{x_0}^{-1}\left[F'(y) - F'(x)\right]\right\| \leqslant f'(\|x - x_0\| + \|y - x\|) - f'(\|x - x_0\|), \tag{13}$$

for all $x, y \in B(x_0, R)$ such that $\|x - x_0\| + \|y - x\| < R$ and satisfies the following conditions:

h1) $f(0) > 0$, $f'(0) = -1$;

h2) f' is convex and strictly increasing;

h3) $f(t) = 0$ for some $t \in (0, R)$.

We also need of the following condition on the majorant condition f which will be considered to hold only when explicitly stated.

h4) $f(t) < 0$ for some $t \in (0, R)$.

Remark 2. *Since $f(0) > 0$ and f is continuous then condition* **h4** *implies condition* **h3**.

Theorem 7. *Let $\mathbb{X}$, $\mathbb{Y}$ be Banach spaces, $\mathbb{X}$ reflexive, $\Omega \subseteq \mathbb{X}$ an open set, $F : \Omega \to \mathbb{Y}$ a continuously Fréchet differentiable function, $C \subset \mathbb{Y}$ a nonempty closed convex cone, $R > 0$ and $f : [0, R) \to \mathbb{R}$ a continuously differentiable function. Suppose that $x_0 \in \Omega$, F satisfies the Robson's condition at x_0, f is a majorant function for F at x_0 and*

$$\left\|T_{x_0}^{-1}[-F(x_0)]\right\| \leqslant f(0)\,. \tag{14}$$

Let

$$\beta := \sup\{-f(t) \ : \ t \in [0, R)\}, \qquad \bar{\tau} := \sup\{t \in [0, R) \, : \, f(t) < 0\}.$$

Take $0 \leq \rho < \beta/2$ and define

$$\kappa_\rho := \sup_{\rho < t < R} \frac{-(f(t) + 2\rho)}{|f'(\rho)|\,(t - \rho)}, \qquad \lambda_\rho := \sup\{t \in [\rho, R) : \kappa_\rho + f'(t) < 0\}, \qquad \tilde{\theta}_\rho := \frac{\kappa_\rho}{2 - \kappa_\rho}. \tag{15}$$

Then for any $\theta \in [0, \tilde{\theta}_\rho]$ *and* $z_0 \in B(x_0, \rho)$*, the sequence* $\{z_k\}$ *generated by inexact Newton's Method for solving the inclusion* $F(x) \in C$ *with starting point* z_0 *and residual relative error tolerance* $\theta : z_{k+1} = z_k + d_k$,

$$d_k \in \arg\min_{d \in \mathbb{X}} \left\{ \|d\| \ : \ F(z_k) + F'(z_k)d + r_k \in C \right\}, \qquad \max_{w \in \{-r_k,\ r_k\}} \left\|T_{x_0}^{-1} w\right\| \leq \theta \left\|T_{x_0}^{-1}[-F(z_k)]\right\|,$$

for all $k = 0, 1, ...$*, is well defined, for any particular choice of each* d_k*,*

$$\|T_{z_0}^{-1}[-F(z_k)]\| \leq \left(\frac{1+\theta^2}{2}\right)^k [f(0) + 2\rho], \tag{16}$$

the sequence $\{z_k\}$ *is contained in* $B(z_0, \lambda_\rho)$ *and converges to a point* $x_* \in B[x_0, \lambda_\rho]$ *such that* $F(x_*) \in C$*. Moreover, if*

h5) $\lambda_\rho < R - \rho$,

then the sequence $\{z_k\}$ *satisfies, for* $k = 0, 1, \ldots$,

$$\|z_k - z_{k+1}\| \leq \frac{1+\theta}{1-\theta} \left[\frac{1+\theta}{2} \frac{D^- f'(\lambda_\rho + \rho)}{|f'(\lambda_\rho + \rho)|} \|z_k - z_{k-1}\| + \theta \frac{2|f'(\rho)| + f'(\lambda_\rho + \rho)}{|f'(\lambda_\rho + \rho)|} \right] \|z_k - z_{k-1}\|. \tag{17}$$

If, additionally, $0 \leq \theta < [-2(\kappa_\rho + 1) + \sqrt{4(\kappa_\rho + 1)^2 + \kappa_\rho(4 + \kappa_\rho)}]/[4 + \kappa_\rho]$ *then* $\{z_k\}$ *converges Q-linearly as follows*

$$\limsup_{k \to \infty} \frac{\|x_* - z_{k+1}\|}{\|x_* - z_k\|} \leq \frac{1+\theta}{1-\theta} \left[\frac{1+\theta}{2} + \frac{2\theta}{\kappa_\rho} \right], \qquad k = 0, 1, \ldots. \tag{18}$$

Remark 3. *In Theorem 7 if* $\theta = 0$ *we obtain the exact Newton method as in [12] and its convergence properties. Now, taking* $\theta = \theta_k$ *in each iteration and letting* θ_k *goes to zero as k goes to infinity, inequality* (17) *implies that the sequence* $\{z_k\}$ *converges to the solution of* (1) *with asymptotic superlinear rate. If* $C = \{0\}$ *we obtain the inexact Newton method as in [15] and its convergence properties are similar.*

Henceforth we assume that the assumption on Theorem 7 holds, except **h5** which will be considered to hold only when explicitly stated.

2.1 Preliminary results

We will first prove Theorem 7 for the case $\rho = 0$ and $z_0 = x_0$. In order to simplify the notation in the case $\rho = 0$, we will use κ, λ and θ instead of κ_0, λ_0 and $\tilde{\theta}_0$ respectively:

$$\kappa := \sup_{0<t<R} \frac{-f(t)}{t}, \qquad \lambda := \sup\{t \in [0, R) : \kappa + f'(t) < 0\}, \qquad \tilde{\theta} := \frac{\kappa}{2-\kappa}. \tag{19}$$

2.1.1 The majorant function

In this subsection we will prove the main results about the majorant function. Define

$$t_* := \min f^{-1}(\{0\}), \qquad \bar{t} := \sup \left\{ t \in [0, R) : f'(t) < 0 \right\}.$$

Then we have the following results:

Proposition 8. *For* κ, λ, θ *as in* (19) *it holds that* $0 < \kappa < 1$, $0 < \theta < 1$ *and* $t_* < \lambda \leq \bar{t} \leq \bar{\tau}$*. Moreover,* $f'(t) + \kappa < 0$*, for* $t \in [0, \lambda)$ *and* $\inf_{0 \leq t < R}(f(t) + \kappa t) = \lim_{t \to \lambda_-}(f(t) + \kappa t) = 0$.

Proof. See [15, Proposition 2.4]. □

Proposition 9. *The majorant function f has a smallest root $t_* \in (0, R)$, is strictly convex and*

$$f(t) > 0, \quad f'(t) < 0, \qquad t < t - f(t)/f'(t) < t_*, \qquad\qquad t \in [0, t^*).$$

Moreover, $f'(t_*) \leqslant 0$ and the following equivalence holds:

$$f'(t_*) < 0 \iff \exists\, t \in (t_*, R),\ f(t) < 0.$$

If, additionally, f satisfies condition **h4** *then the following statements hold:*

i) $f'(t) < 0$ *for any* $t \in [0, \bar{t})$*;*

ii) $0 < t_* < \bar{t} \leq R$*;*

iii) $\beta = -\lim_{t \to \bar{t}_-} f(t), \qquad 0 < \beta < \bar{t}$*;*

iv) If $0 \leq \rho < \beta/2$ *then* $\rho < \bar{t}/2 < \bar{t}$ *and* $f'(\rho) < 0$.

Proof. See [15, Propositions 2.3 and 5.2] and [14, Proposition 3]. □

Take $0 \leq \theta$ and $0 \leq \varepsilon$. We will need of the following auxiliary mapping $n_\theta : [0, \bar{t}\,) \times [0, \infty) \to \mathbb{R} \times \mathbb{R}$,

$$n_\theta(t, \varepsilon) := \left(t - (1+\theta)\frac{f(t) + \varepsilon}{f'(t)},\ \varepsilon + 2\theta(f(t) + \varepsilon) \right), \tag{20}$$

and the following auxiliary set

$$\mathcal{A} := \{(t, \varepsilon) \in \mathbb{R} \times \mathbb{R}\ :\ 0 \leq t < \lambda,\ 0 \leq \varepsilon \leq \kappa t,\ 0 < f(t) + \varepsilon\}. \tag{21}$$

Lemma 10. *If $0 \leq \theta \leq \tilde{\theta}$, $(t, \varepsilon) \in \mathcal{A}$ and $(t_+, \varepsilon_+) := n_\theta(t, \varepsilon)$, that is,*

$$t_+ := t - (1+\theta)\frac{f(t) + \varepsilon}{f'(t)}, \qquad\qquad \varepsilon_+ := \varepsilon + 2\theta(f(t) + \varepsilon),$$

then $n_\theta(t, \varepsilon) \in \mathcal{A}$, $t < t_+$ and $\varepsilon \leq \varepsilon_+$. Moreover,

$$f(t_+) + \varepsilon_+ < \left(\frac{1 + \theta^2}{2}\right)(f(t) + \varepsilon).$$

Proof. See [15, Lemma 4.2]. □

Define the *linearization error* of the majorant function associated to F as follows

$$e_f(v, t) := f(v) - [f(t) + f'(t)(v - t)], \qquad t, s \in [0, R). \tag{22}$$

As we will see, in the next section, the inearization error of the majorant function bounds the linearization error of the function F. We will need the following result about the majorant function.

Lemma 11. *If $0 \leq b \leq t$, $0 \leq a \leq s$ and $t + s < R$, then there holds:*

$$e_f(a + b, b) \leq \max\left\{ e_f(t + s, t),\ \frac{1}{2}\frac{f'(t+s) - f'(t)}{s}\ a^2 \right\}, \qquad s \neq 0.$$

Proof. See [15, Lemma 3.3]. □

2.1.2 Relationships between the majorant and nonlinear functions

In this section, we will present the main relationships between the majorant function f and the nonlinear function F that we need for proving Theorem 7. Note that Robson's condition, namely, $\operatorname{rge} T_{x_0} = \mathbb{Y}$ implies that $\operatorname{dom} T_{x_0}^{-1} = \mathbb{Y}$.

Proposition 12. *If* $\|x - x_0\| \leqslant t < \bar{t}$ *then* $dom\,[T_x^{-1}F'(x_0)] = \mathbb{X}$ *and there holds*

$$\left\|T_x^{-1}F'(x_0)\right\| \leqslant -1/f'(t).$$

As a consequence, $rge\,T_x = \mathbb{Y}$*.*

Proof. See [12, Proposition 12]. □

Newton's iteration at a point $x \in \Omega$ happens to be a solution of the linearization of the inclusion $F(y) \in C$ at such a point, namely, a solution of the linear inclusion $F(x) + F'(x)(x - y) \in C$. So, we study the linearization error of F at a point in Ω

$$E_F(y, x) := F(y) - \left[F(x) + F'(x)(y - x)\right], \qquad y,\, x \in \Omega. \tag{23}$$

We will bound this error by (22), the error in the linearization on the majorant function associated to F.

Lemma 13. *If* $x, y \in \mathbb{X}$ *and* $\|x - x_0\| + \|y - x\| < R$ *then*

$$\|T_{x_0}^{-1}E_F(y, x)\| \leq e_f(\|x - x_0\| + \|y - x\|, \|x - x_0\|).$$

Proof. As $x, y \in B(x_0, R)$ and the ball is convex $x + \tau(y - x) \in B(x_0, R)$, for all $\tau \in [0, 1]$. Since, by assumption, $\operatorname{rge} T_{x_0} = \mathbb{Y}$ we obtain that $\operatorname{dom} T_{x_0}^{-1} = \mathbb{Y}$. Thus, using that $F'(z)$ is a linear mapping for each $z \in \mathbb{X}$, we conclude

$$\left\|T_{x_0}^{-1}\left([F'(x + \tau(y - x)) - F'(x)](y - x)\right)\right\| \leq \left\|T_{x_0}^{-1}[F'(x + \tau(y - x)) - F'(x)]\right\| \|y - x\|,$$

for all $\tau \in [0, 1]$. Hence, as f is a majorant function for F at x_0, using (13) and last inequality we have

$$\left\|T_{x_0}^{-1}\left([F'(x + \tau(y - x)) - F'(x)](y - x)\right)\right\| \leqslant \left[f'\left(\|x - x_0\| + \tau\,\|y - x\|\right) - f'\left(\|x - x_0\|\right)\right] \|y - x\|,$$

for all $\tau \in [0, 1]$. Thus, since $\operatorname{dom} T_{x_0}^{-1} = \mathbb{Y}$, we apply Lemma 5 with $U = T_{x_0}^{-1}$ and the functions $G(\tau)$ and $g(\tau)$ equals to the expressions in the last inequality, in parentheses on the left hand side and on the right hand side, respectively, obtaining

$$\left\|T_{x_0}^{-1}\int_0^1 [F'(x + \tau(y - x)) - F'(x)](y - x)\, d\tau\right\| \leqslant \int_0^1 \left[f'\left(\|x - x_0\| + \tau\,\|y - x\|\right) - f'\left(\|x - x_0\|\right)\right] \|y - x\|\, d\tau,$$

which, after performing the integration of the right hand side, taking into account the definition of $e_f(v, t)$ in (22) and that (23) is equivalent to

$$E_F(y, x) = \int_0^1 [F'(x + \tau(y - x)) - F'(x)](y - x)\, d\tau,$$

yields the desired inequality. □

Lemma 14. *If* $x, y \in \mathbb{X}$ *and* $\|x - x_0\| + \|y - x\| < R$ *then*

$$\|T_{x_0}^{-1}[-E_F(y, x)]\| \leq e_f(\|x - x_0\| + \|y - x\|, \|x - x_0\|).$$

Proof. To prove this lemma we follow the same arguments used in the proof of Lemma 13, by taking into account Remark 1. ☐

Corollary 15. *If $x, y \in \mathbb{X}$, $\|x - x_0\| \le t$, $\|y - x\| \le s$ and $s + t < R$ then*

$$\max\left\{\left\|T_{x_0}^{-1}\left[-E_F(y,x)\right]\right\|,\ \|T_{x_0}^{-1}E_F(y,x)\|\right\} \le \max\left\{e_f(t+s,t),\ \frac{1}{2}\frac{f'(s+t)-f'(t)}{s}\|y-x\|^2\right\}, \quad s \ne 0.$$

Proof. The results follows by direct combination of the Lemmas 13, 14 and 11 by taking $b = \|x - x_0\|$ and $a = \|y - x\|$. ☐

Lemma 16. *If $x \in \mathbb{X}$ and $\|x - x_0\| \le t < R$ then $\|T_{x_0}^{-1}F'(x)\| \le 2 + f'(t)$.*

Proof. First of all, we use Definition of sublinear mapping in (9) to obtain

$$T_{x_0}^{-1}F'(x) \supseteq T_{x_0}^{-1}[F'(x) - F'(x_0)] + T_{x_0}^{-1}F'(x_0).$$

Hence, taking into account properties of the norm, we conclude from above inclusion that

$$\left\|T_{x_0}^{-1}F'(x)\right\| \le \left\|T_{x_0}^{-1}[F'(x) - F'(x_0)]\right\| + \left\|T_{x_0}^{-1}F'(x_0)\right\|.$$

Since $T_{x_0}^{-1}F'(x_0) \supseteq F'(x_0)^{-1}F'(x_0)$ we have $\|\,\left\|T_{x_0}^{-1}F'(x_0)\right\|\,\| \le 1$. Thus, using assumption (13), the last inequality becomes

$$\left\|T_{x_0}^{-1}F'(x)\right\| \le f'(\|x - x_0\|) - f'(0) + 1.$$

Therefore, assumptions **h1**, **h2** and the last inequality imply the statement of the lemma. ☐

The inequality in the next proposition will be used to show that inexact Newton's method is robust with respect to the initial iterate.

Proposition 17. *For any $y \in B(x_0, R)$, the following inequality holds:*

$$\left\|T_{x_0}^{-1}[-F(y)]\right\| \le f(\|y - x_0\|) + 2\|y - x_0\|.$$

Proof. To prove see [12, Proposition 16]. ☐

3 Convergence analysis of the inexact Newton Method

In this section we will prove Theorem 7. Before proving Theorem 7, we need to study the inexact Newton's iteration, associated to the function F, and prove Theorem 7 for the case $\rho = 0$ and $z_0 = x_0$.

3.1 The inexact Newton iteration

The outcome of an inexact Newton iteration is any point satisfying some error tolerance. Hence, instead of a mapping for inexact Newton iteration, we shall deal with a *family* of mappings, describing all possible inexact iterations. Before defining the inexact Newton iteration mapping, we need to define the *inexact Newton's step mapping, $D_{F,C,\theta} : B(x_0, \bar{t}) \rightrightarrows \mathbb{X}$,*

$$D_{F,C,\theta}(x) := \operatorname*{arg\,min}_{d \in \mathbb{X}}\left\{\|d\| \ :\ F(x) + F'(x)d + r \in C\right\}; \qquad \max_{w \in \{-r,\ r\}}\left\|T_{x_0}^{-1}w\right\| \le \theta\left\|T_{x_0}^{-1}[-F(x)]\right\|, \quad (24)$$

associated to F, C and θ. Since $\mathbb{X}$ is reflexive, second part of Proposition 12 guarantees, in particular, that exact Newton's step $D_{F,C,0}(x)$ is nonempty, for each $x \in B(x_0, \bar{t})$. Since $D_{F,C,0}(x) \subseteq D_{F,C,\theta}(x)$, we conclude $D_{F,C,\theta}(x) \ne \varnothing$, for $x \in B(x_0, \bar{t})$. Therefore, for $0 \le \theta \le \tilde{\theta}$, we can define $\mathcal{N}_\theta$ the *family of inexact Newton iteration mapping*, $N_{F,C,\theta} : B(x_0, \bar{t}) \rightrightarrows \mathbb{X}$,

$$N_{F,C,\theta}(x) := x + D_{F,C,\theta}(x). \quad (25)$$

One can apply a *single* Newton's iteration on any $x \in B(x_0, \bar{t})$ to obtain the set $N_{F,C,\theta}(x)$, which may not is contained to $B(x_0, \bar{t})$, or even may not is in the domain of F. Therefore, this is enough to guarantee the well-definedness of only one iteration. To ensure that inexact Newtonian iteration mapping may be repeated indefinitely, we need some additional results. First, define some subsets of $B(x_0, \bar{t})$ in which, as we shall prove, inexact Newton iteration mappings (25) are "well behaved". Define

$$K(t, \varepsilon) := \left\{ x \in \mathbb{X} \ : \ \|x - x_0\| \le t, \ \|T_{x_0}^{-1}[-F(x)]\| \le f(t) + \varepsilon \right\}, \tag{26}$$

and

$$\mathcal{K} := \bigcup_{(t,\varepsilon) \in \mathcal{A}} K(t, \varepsilon). \tag{27}$$

Proposition 18. *Take $0 \le \theta \le \tilde{\theta}$ and $N_{F,C,\theta} \in \mathcal{N}_\theta$. Then, for any $(t, \varepsilon) \in \mathcal{A}$ and $x \in K(t, \varepsilon)$*

$$\|y - x\| \le t_+ - t, \tag{28}$$

where $y \in N_{F,C,\theta}(x)$ and t_+ is the first component of the function $n_\theta(t, \varepsilon)$ defined in (20)*. Moreover,*

$$N_{F,C,\theta}(K(t, \varepsilon)) \subset K(n_\theta(t, \varepsilon)). \tag{29}$$

As a consequence,

$$n_\theta(\mathcal{A}) \subset \mathcal{A}, \qquad N_{F,C,\theta}(\mathcal{K}) \subset \mathcal{K}. \tag{30}$$

Proof. Take $0 \le \theta$, $(t, \varepsilon) \in \mathcal{A}$ and $x \in K(t, \varepsilon)$. Thus, the definitions of the sets $\mathcal{A}$ in (21), $K(t, \varepsilon)$ in (26) together with Lemma 10 imply that

$$\|x - x_0\| \le t < \bar{t}, \qquad \|T_{x_0}^{-1}[-F(x)]\| \le f(t) + \varepsilon, \qquad t - (1 + \theta)\frac{f(t) + \varepsilon}{f'(t)} < \lambda \le R. \tag{31}$$

Take $y \in N_{F,C,\theta}(x)$ and r as in (24). Using the third property of convex process in (9), we have

$$T_x^{-1}[-F(x) - r] \supseteq T_x^{-1}[-F(x)] + T_x^{-1}[-r].$$

Applying Lemma 6 in each term in the right hand side of last inclusion, one with $w = -r$, $z = x$ and $v = x_0$, and the other one with $w = -F(x)$, $z = x$ and $v = x_0$, we obtain

$$T_x^{-1}[-F(x) - r] \supseteq T_x^{-1}F'(x_0)T_{x_0}^{-1}[-F(x)] + T_x^{-1}F'(x_0)T_{x_0}^{-1}[-r].$$

Hence, taking norm in both sides of last inclusion and using the properties of the norm yields

$$\left\|T_x^{-1}[-F(x) - r]\right\| \le \left\|T_x^{-1}F'(x_0)\right\| \left\|T_{x_0}^{-1}[-F(x)]\right\| + \left\|T_x^{-1}F'(x_0)\right\| \left\|T_{x_0}^{-1}[-r]\right\|.$$

Considering that $y - x \in D_{F,C,\theta}(x)$, we obtain that $\|y - x\| = \|T_x^{-1}[-F(x) - r]\|$. Thus, combining last inequality with Proposition 12 and the third inequality in (31), after some manipulation taking into account (24), we have

$$\|y - x\| \le -(1 + \theta)\frac{f(t) + \varepsilon}{f'(t)}, \tag{32}$$

which, using definition of t_+, is equivalent to (28).

Since $\|y - x_0\| \le \|y - x\| + \|x - x_0\|$, thus (32), the first and the last inequality in (31) give

$$\|y - x_0\| \le t - (1 + \theta)\frac{f(t) + \varepsilon}{f'(t)} < \lambda \le R. \tag{33}$$

On the other hand, the linearization error in (23) and the third property of convex process in (9) imply

$$T_{x_0}^{-1}[-F(y)] \supseteq T_{x_0}^{-1}[-E_F(y, x)] + T_{x_0}^{-1}[-F(x) - F'(x)(y - x)].$$

Thus, taking norm in both sides of last inclusion and using the triangle inequality we obtain

$$\|T_{x_0}^{-1}[-F(y)\| \leq \left\|T_{x_0}^{-1}[-E_F(y,x)]\right\| + \left\|T_{x_0}^{-1}[-F(x) - F'(x)(y-x)]\right\|.$$

Since $y \in N_{F,C,\theta}(x)$ we have $T_{x_0}^{-1}[r] \subset T_{x_0}^{-1}[-F(x)-F'(x)(y-x)]$, where r satisfies $F(x)+F'(x)(y-x)+r \in C$ and (24). Then, last inequality implies

$$\|T_{x_0}^{-1}[-F(y)\| \leq \left\|T_{x_0}^{-1}[-E_F(y,x)]\right\| + \theta\left\|T_{x_0}^{-1}[-F(x)]\right\|.$$

The second term in the right hand side of last inequality is bound by third inequality in (31). Thus, letting $s = -(1+\theta)(f(t)+\varepsilon)/f'(t)$, using (32), first and last inequality (31), we can apply Corollary 15 to conclude that

$$\|T_{x_0}^{-1}[-F(y)\| \leq e_f\left(t - (1+\theta)\frac{f(t)+\varepsilon}{f'(t)}, t\right) + \theta(f(t)+\varepsilon).$$

Therefore, combining the last inequality with the definition in (22), we easily obtain that

$$\|T_{x_0}^{-1}[-F(y)]\| \leq f\left(t - (1+\theta)\frac{f(t)+\varepsilon}{f'(t)}\right) + \varepsilon + 2\theta(f(t)+\varepsilon).$$

Finally, (33), last inequality, definitions (20) and (26) proof that the inclusion (29) holds.

The inclusions in (30) is an immediate consequence of Lemma 10, (29) and the definitions in (21) and (27). Thus, the proof of the proposition is concluded. □

3.2 Convergence analysis

In this section we will proof Theorem 7. First we will show that the sequence generated by inexact Newton method is well behaved with respect to the set defined in (26).

Theorem 19. *Take $0 \leq \theta \leq \tilde{\theta}$ and $N_{F,C,\theta} \in \mathcal{N}_\theta$. For any $(t_0, \varepsilon_0) \in \mathcal{A}$ and $y_0 \in K(t_0, \varepsilon_0)$ the sequences*

$$y_{k+1} \in N_{F,C,\theta}(y_k), \qquad (t_{k+1}, \varepsilon_{k+1}) = n_\theta(t_k, \varepsilon_k), \qquad k = 0, 1, \ldots, \tag{34}$$

are well defined,

$$y_k \in K(t_k, \varepsilon_k), \qquad (t_k, \varepsilon_k) \in \mathcal{A} \qquad k = 0, 1, \ldots, \tag{35}$$

the sequence $\{t_k\}$ is strictly increasing and converges to some $\tilde{t} \in (0, \lambda]$, the sequence $\{\varepsilon_k\}$ is non-decreasing and converges to some $\tilde{\varepsilon} \in [0, \kappa\lambda]$,

$$\left\|T_{x_0}^{-1}[-F(y_k)]\right\| \leq f(t_k) + \varepsilon_k \leq \left(\frac{1+\theta^2}{2}\right)^k (f(t_0) + \varepsilon_0), \quad k = 0, 1, \ldots, \tag{36}$$

$\{y_k\}$ is contained in $B(x_0, \lambda)$, converges to a point $x_* \in B[x_0, \lambda]$ such that $F(x_*) \in C$, and satisfies

$$\|y_{k+1} - y_k\| \leq t_{k+1} - t_k, \qquad \|x_* - y_k\| \leq \tilde{t} - t_k, \qquad k = 0, 1, \ldots. \tag{37}$$

Moreover, if

h5') *$\lambda < R$,*

then the sequence $\{y_k\}$ satisfies

$$\|y_k - y_{k+1}\| \leq \frac{1+\theta}{1-\theta}\left[\frac{1+\theta}{2}\frac{D^- f'(\lambda)}{|f'(\lambda)|}\|y_k - y_{k-1}\| + \theta\,\frac{2+f'(\lambda)}{|f'(\lambda)|}\right]\|y_k - y_{k-1}\|, \qquad k = 0, 1, \ldots. \tag{38}$$

If, additionally, $0 \leq \theta < -2(\kappa+1) + \sqrt{4(\kappa+1)^2 + \kappa(4+\kappa)}/(4+\kappa)$ then $\{y_k\}$ converges Q-linearly as follows

$$\limsup_{k\to\infty} \frac{\|x_* - y_{k+1}\|}{\|x_* - y_k\|} \leq \frac{1+\theta}{1-\theta}\left[\frac{1+\theta}{2} + \frac{2\theta}{\kappa}\right], \qquad k = 0, 1, \ldots. \tag{39}$$

Proof. Since $0 \le \theta \le \tilde{\theta}$, $(t_0, \varepsilon_0) \in \mathcal{A}$ and $y_0 \in K(t_0, \varepsilon_0)$, the well definition of the sequences $\{(t_k, \varepsilon_k)\}$ and $\{y_k\}$, as defined in (34), follow from the two last inclusions (30) in Proposition 18. Moreover, since (35) holds for $k = 0$, using the first inclusion in Proposition 18, first inclusion in (30) and induction on k, we conclude that (35) holds for all k. The first inequality in (37) follows from (28) in Proposition 18, (34) and (35), while the first inequality in (36) follows from (35) and the definition of $K(t, \varepsilon)$ in (26).

The definition of $\mathcal{A}$ in (21) implies $\mathcal{A} \subset [0, \lambda) \times [0, \kappa\lambda)$. Therefore, using (35) and the definition of $K(t, \varepsilon)$ we have

$$t_k \in [0, \lambda), \quad \varepsilon_k \in [0, \kappa\lambda), \quad y_k \in B(x_0, \lambda), \qquad k = 0, 1, \ldots.$$

Using (21) and Lemma 10 we conclude that $\{t_k\}$ is strictly increasing, $\{\varepsilon_k\}$ is non-decreasing and the second equality in (36) holds for all k. Therefore, in view of the first two above inclusions, $\{t_k\}$ and $\{\varepsilon_k\}$ converge, respectively, to some $\tilde{t} \in (0, \lambda]$ and $\tilde{\varepsilon} \in [0, \kappa\lambda]$. The convergence of $\{t_k\}$ to $\tilde{t}$, together with the first inequality in (37) and the inclusion $y_k \in B(x_0, \lambda)$ implies that y_k converges to some $x_* \in B[x_0, \lambda]$ and that the second inequality on (37) holds for all k. Moreover, taking the limit in (36), as k goes to $+\infty$, we conclude that

$$\lim_{k \to +\infty} \left\| T_{x_0}^{-1}[-F(y_k)] \right\| = 0.$$

Thus, there exists $\{d_k\} \subset \mathbb{X}$ such that $d_k \in T_{x_0}^{-1}[-F(y_k)]$, for all $k = 0, 1, \ldots$, with $\lim_{k \to +\infty} d_k = 0$. Since $d_k \in T_{x_0}^{-1}[-F(y_k)]$, for all $k = 0, 1, \ldots$, the definition 12 implies that $F'(x_0)d_k + F(y_k) \in C$, for all $k = 0, 1, \ldots$. Hence, letting k goes to $+\infty$ in last inclusion and taking into account that C is closed and $\{y_k\}$ converges to x_*, we conclude that $F(x_*) \in C$.

We are going to prove (38). Since $y_{k+1} \in N_{F,C,\theta}(y_k)$, for k= 0,1, ... , we have

$$\|y_{k+1} - y_k\| = \|T_{y_k}^{-1}[-F(y_k) - r_k]\|, \qquad \max_{w \in \{-r_k,\ r_k\}} \left\| T_{x_0}^{-1} w \right\| \le \theta \left\| T_{x_0}^{-1}[-F(y_k)] \right\|. \tag{40}$$

The third property in (9) implies $T_{y_k}^{-1}[-F(y_k) - r_k] \supseteq T_{y_k}^{-1}[-F(y_k)] + T_{y_k}^{-1}[-r_k]$. Then applying twice Lemma 6, one with $z = y_k$, $v = x_0$ and $w = -F(y_k)$ and, the other one, with $z = y_k$, $v = x_0$ and $w = -r_k$, we obtain that

$$T_{y_k}^{-1}[-F(y_k) - r_k] \supseteq T_{y_k}^{-1} F'(x_0) T_{x_0}^{-1}[-F(y_k)] + T_{y_k}^{-1} F'(x_0) T_{x_0}^{-1}[-r_k].$$

Combining last inclusion with (40) and properties of the norm we concluded, after some algebra, that

$$\|y_{k+1} - y_k\| \le (1 + \theta) \left\| T_{y_k}^{-1} F'(x_0) \right\| \left\| T_{x_0}^{-1}[-F(y_k)] \right\|. \tag{41}$$

Using (23), the third property in (9) and triangular inequality, after some manipulation, we have

$$\left\| T_{x_0}^{-1}[-F(y_k)] \right\| \le \left\| T_{x_0}^{-1}[-E_F(y_k, y_{k-1})] \right\| + \left\| T_{x_0}^{-1}[-F(y_{k-1}) - F'(y_{k-1})(y_k - y_{k-1})] \right\|. \tag{42}$$

On the other hand, because $y_k \in N_{F,C,\theta}(y_{k-1})$ we have $T_{x_0}^{-1}[r_{k-1}] \subset T_{x_0}^{-1}[-F(y_{k-1}) - F'(y_{k-1})(y_k - y_{k-1})]$, where r_{k-1} satisfies

$$\left\| T_{x_0}^{-1} r_{k-1} \right\| \le \theta \left\| T_{x_0}^{-1}[-F(y_{k-1})] \right\|.$$

Therefore, we have

$$\left\| T_{x_0}^{-1}[-F(y_{k-1}) - F'(y_{k-1})(y_k - y_{k-1})] \right\| \le \theta \left\| T_{x_0}^{-1}[-F(y_{k-1})] \right\|, \tag{43}$$

which combined with the inequalities in (41) and (42) yields

$$\|y_{k+1} - y_k\| \le (1 + \theta) \left\| T_{y_k}^{-1} F'(x_0) \right\| \Big[\left\| T_{x_0}^{-1}[-E_F(y_k, y_{k-1})] \right\| + \theta \left\| T_{x_0}^{-1}[-F(y_{k-1})] \right\| \Big]. \tag{44}$$

Using again (23), the third property in (9) and triangular inequality, we obtain after some algebra that

$$\left\| T_{x_0}^{-1}[-F(y_{k-1}) \right\| \le \left\| T_{x_0}^{-1} E_F(y_k, y_{k-1}) \right\| + \left\| T_{x_0}^{-1}[-F(y_k)] \right\| + \left\| T_{x_0}^{-1} F'(y_{y-1})(y_k - y_{k-1}) \right\|.$$

Combining the last inequality with the inequalities in (42) and (43) we concluded that

$$\|T_{x_0}^{-1}[-F(y_{k-1})\| \leq \frac{1}{1-\theta}\left[\|T_{x_0}^{-1}[E_F(y_k, y_{k-1})]\| + \|T_{x_0}^{-1}[-E_F(y_k, y_{k-1})]\| + \|T_{x_0}^{-1}F'(y_{y-1})(y_k - y_{k-1})\|\right].$$

Inequality in (44) combined with last inequality becomes

$$\|y_{k+1} - y_k\| \leq \frac{1+\theta}{1-\theta}\left\|T_{y_k}^{-1}F'(x_0)\right\|\Big[\left\|T_{x_0}^{-1}[-E_F(y_k, y_{k-1})]\right\| + \\ \theta\left(\left\|T_{x_0}^{-1}[E_F(y_k, y_{k-1})]\right\| + \left\|T_{x_0}^{-1}F'(y_{k-1})(y_k - y_{k-1})\right\|\right)\Big].$$

Therefore, combining last inequality with Proposition 12, Lemma 16 and Corollary 15 with $x = y_{k-1}$, $y = y_k$, $s = t_k - t_{k-1}$ and $t = t_{k-1}$, we have

$$\|y_k - y_{k+1}\| \leq \frac{1+\theta}{1-\theta}\frac{1}{|f'(t_k)|}\left[\frac{1+\theta}{2}\frac{f'(t_k) - f'(t_{k-1})}{t_k - t_{k-1}}\|y_{k-1} - y_k\| + \theta[2 + f'(t_{k-1})]\right]\|y_{k-1} - y_k\|, \quad (45)$$

for $k = 0, 1, \ldots$. Since $\|y_{k-1} - y_k\| \leq t_k - t_{k-1}$, see (37), $f' < -\kappa < 0$ in $[0, \lambda)$, (39) follows from last inequality. Using **h5'** and Proposition 1 and taking into account that $|f'|$ is decreasing in $[0, \lambda]$, f' is increasing in $[0, \lambda]$ and $\{t_k\} \subset [0, \lambda]$, we obtain that (38) follows from above inequality.

For concluding the proof, it remains to prove that $\{y_k\}$ converges Q-linearly as in (39). First note that $\|y_{k-1} - y_k\| \leq t_k - t_{k-1}$ and $f'(t_{k-1}) \leq f'(t_k) < 0$. Thus, we conclude from (46) that

$$\|y_k - y_{k+1}\| \leq \frac{1+\theta}{1-\theta}\left[\frac{1+\theta}{2} + \frac{2\theta}{\kappa}\right]\|y_{k-1} - y_k\|, \qquad k = 0, 1, \ldots. \quad (46)$$

which, from Proposition 2, implies that (39) holds. Since $0 \leq \theta < -2(\kappa+1) + \sqrt{4(\kappa+1)^2 + \kappa(4+\kappa)}/(4+\kappa)$, the quantity in the right hand side of (39) is less than one. Hence $\{y_k\}$ converges Q-linearly, which conclude the proof. □

Proposition 20. *Let $R > 0$ and $f : [0, R) \to \mathbb{R}$ a continuously differentiable function. Suppose that $x_0 \in \Omega$, f is a majorant function for F at x_0 and satisfies* **h4**. *If $0 \leq \rho < \beta/2$, then for any $z_0 \in B(x_0, \rho)$ the scalar function $g : [0, R - \rho) \to \mathbb{R}$, defined by*

$$g(t) := \frac{-1}{f'(\rho)}[f(t+\rho) + 2\rho], \quad (47)$$

is a majorant function for F at z_0 and also satisfies condition **h4**.

Proof. To prove see Proposition 17 of [12]. □

Proof of Theorem 7. First we will prove Theorem 7 with $\rho = 0$ and $z_0 = x_0$. Note that, from the definition in (19), we have

$$\kappa_0 = \kappa, \quad \lambda_0 = \lambda, \quad \tilde{\theta}_0 = \tilde{\theta}.$$

The assumption (14) implies that $x_0 \in K(0, 0)$. Since $(t_0, \varepsilon_0) = (0, 0) \in \mathcal{A}$ and $y_0 = x_0 \in K(0, 0)$, we apply Theorem 19 with $z_k = y_k$, for $k = 0, 1, \ldots$, to conclude that Theorem 7 holds for $\rho = 0$ and $z_0 = x_0$.

We are going to prove the general case. From item *iv* of Proposition 9 we have $\rho < \bar{t}$, which implies that $\|z_0 - x_0\| < \rho < \bar{t}$. Thus, we can apply Proposition 12 to obtain

$$\left\|T_{z_0}^{-1}F'(x_0)\right\| \leq \frac{-1}{f'(\rho)}. \quad (48)$$

Moreover, the point z_0 satisfies the Robinson's condition, namely,

$$\operatorname{rge} T_{z_0} = \mathbb{Y}.$$

Then, using Lemma 6, property of the norm, (48) and Proposition 17 with $y = z_0$ we have

$$\left\|T_{z_0}^{-1}[-F(z_0)]\right\| \leq \left\|T_{z_0}^{-1}F'(x_0)\right\| \left\|T_{x_0}^{-1}[-F(z_0)]\right\| \\ \leq \frac{-1}{f'(\rho)}[f(\|z_0 - x_0\|) + 2\|z_0 - x_0\|].$$

Since $f' \geq -1$, the function $t \mapsto f(t) + 2t$ is (strictly) increasing. Thus, combining this fact with the last inequality, the inequality $\|z_0 - x_0\| < \rho$ and (47) we conclude that

$$\left\|T_{z_0}^{-1}[-F'(z_0)]\right\| \leq g(0).$$

Proposition 20 implies that g, defined in (47), is a majorant function for F at point z_0 and also satisfies condition **h4**. Moreover, (47) and κ_ρ, λ_ρ and $\tilde{\theta}_\rho$ as defined in (15) imply

$$\kappa_\rho = \sup_{0<t<R-\rho} \frac{-g(t)}{t}, \qquad \lambda_\rho = \sup\{t \in [0, R-\rho) : \kappa_\rho + g'(t) < 0\}, \qquad \tilde{\theta}_\rho = \frac{\kappa_\rho}{2 - \kappa_\rho},$$

which are the same as (15) with g instead of f, then we can apply Theorem 19 for F and the majorant function g at point z_0 and $\rho = 0$, to concluding that the sequence $\{z_k\}$ is well defined, remains in $B(z_0, \lambda_\rho)$, satisfies (16) and converges to some $x_* \in B[z_0, \lambda_\rho]$ with $F(x_*) \in C$. Furthermore, since

$$g'(t) = f'(t+\rho)/|f'(\rho)|, \qquad D^-g'(t) = D^-f'(t+\rho)/|f'(\rho)|, \qquad t \in [0, R-\rho),$$

after some algebra, we conclude that inequalities (17) and (18) also hold. Therefore, the proof of theorem is concluded. □

4 Special cases of Newton's method for cone inclusion problems

In this section we will use Theorem 7 to analyze the convergence of the inexact Newton's method for cone inclusion problems under affine invariant Lipschitz condition and in the setting of Smale's α-theory. Up to our knowledge, this is the first time that the inexact Newton method for cone inclusion problems with a relative error tolerance under Lipschitz's condition and Smale's condition are analyzed.

4.1 Under affine invariant Lipschitz condition

In this section we present the convergence analysis of the inexact Newton's method for cone inclusion problems under affine invariant Lipschitz condition.

Theorem 21. *Let $\mathbb{X}$, $\mathbb{Y}$ be Banach spaces, $\mathbb{X}$ reflexive, $\Omega \subseteq \mathbb{X}$ an open set, $F : \Omega \to \mathbb{Y}$ a continuously Fréchet differentiable function, $C \subset \mathbb{Y}$ a nonempty closed convex cone. Suppose that $x_0 \in \Omega$, F satisfies the Robson's condition at x_0 and exist $L > 0$ and $b > 0$ such that $bL < 1/2$, $B(x_0, 1/L) \subset \Omega$ and*

$$\left\|T_{x_0}^{-1}\left[F'(y) - F'(x)\right]\right\| \leq L\|x - y\|, \qquad x,\, y \in B(x_0, 1/L),$$

$$\|T_{x_0}^{-1}F(x_0)\| \leq b, \qquad 0 \leq \theta \leq \frac{1 - \sqrt{2bL}}{1 + \sqrt{2bL}}.$$

Then, the sequence $\{x_k\}$ generated by the inexact Newton method for solving $F(x) \in C$ with starting point x_0 and residual relative error tolerance θ: $x_{k+1} = x_k + d_k$,

$$d_k \in \arg\min_{d \in \mathbb{X}} \left\{\|d\| \ : \ F(z_k) + F'(z_k)d + r_k \in C\right\}, \qquad \max_{w \in \{-r_k,\ r_k\}} \left\|T_{x_0}^{-1}w\right\| \leq \theta \left\|T_{x_0}^{-1}[-F(z_k)]\right\|,$$

for all $k = 0, 1, ...,$ is well defined, for any particular choice of each d_k,

$$\|T_{x_0}^{-1}[-F(x_k)]\| \leq \left(\frac{1+\theta^2}{2}\right)^k b, \qquad k = 0, 1, \ldots,$$

the sequence $\{x_k\}$ is contained in $B(x_0, \lambda)$, converges to a point $x_* \in B[x_0, \lambda]$, where $\lambda := \sqrt{2bL}/L$. Moreover, $\{x_k\}$ satisfies

$$\|x_k - x_{k+1}\| \leq \frac{1+\theta}{1-\theta}\left[\frac{1+\theta}{2}\frac{L}{1-\sqrt{2bL}}\|x_{k-1} - x_k\| + \theta\frac{1+\sqrt{2bL}}{1-\sqrt{2bL}}\right]\|x_{k-1} - x_k\|, \qquad k = 0, 1, \ldots.$$

If, additionally, $0 \leq \theta < \left(-2(2-\sqrt{2bL}) + \sqrt{10bL - 14\sqrt{2bL} + 21}\right)/(5 - \sqrt{2bL})$ then $\{x_k\}$ converges Q-linearly as follows

$$\limsup_{k\to\infty} \frac{\|x_* - x_{k+1}\|}{\|x_* - x_k\|} \leq \frac{1+\theta}{1-\theta}\left[\frac{1+\theta}{2} + \frac{2\theta}{1-\sqrt{2bL}}\right], \qquad k = 0, 1, \ldots.$$

Proof. Take $\tilde{\theta} = (1-\sqrt{2bL})/(1+\sqrt{2bL})$. Since $f : [0, 1/L) \to \mathbb{R}$, defined by $f(t) := (L/2)t^2 - t + b$, is a majorant function for F at point x_0, all result follow from Theorem 7, applied to this particular context. □

Remark 4. *In Theorem 21, if $\theta = 0$ and $C = \{0\}$ then we obtain, [12, Theorem 18] for the exact Newton method and [15, Theorem 6.3] for the inexact Newton method, respectively.*

4.2 Under affine invariant Smale's condition

In this section we present the convergence analysis of the inexact Newton's method for cone inclusion problems under affine invariant Smale's condition. For state the convergence result we need of the following result:

Lemma 22. *Let $\mathbb{X}$ and $\mathbb{Y}$ be a Banach spaces such that $\mathbb{X}$ is reflexive, $\Omega \subseteq \mathbb{X}$ and $F : \Omega \to \mathbb{Y}$ a continuous function, twice continuously differentiable on int Ω. Suppose that F satisfies the Robson's condition at x_0. If there exists $f : [0, R) \to \mathbb{R}$ twice continuously differentiable such that*

$$\left\|T_{x_0}^{-1}F''(x)\right\| \leq f''(\|x - x_0\|),$$

for all $x \in \Omega$ such that $\|x - x_0\| < R$, then F and f satisfy (13).

Proof. See Lemma 20 of [12]. □

Theorem 23. *Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces, $\Omega \subseteq \mathbb{X}$ and $F : \Omega \to \mathbb{Y}$ a continuous function and analytic in int(Ω) and $C \subset \mathbb{Y}$ a nonempty closed convex cone. Suppose that $x_0 \in \Omega$, F satisfies Robson's condition at x_0,*

$$\gamma := \sup_{n>1}\left\|\frac{T_{x_0}^{-1}F^{(n)}(x_0)}{n!}\right\|^{1/(n-1)} < +\infty,$$

$B(x_0, 1/\gamma) \subset \Omega$ and there exists $b > 0$ such that

$$\|T_{x_0}^{-1}[-F(x_0)]\| \leq b, \qquad b\gamma < 3 - 2\sqrt{2}, \qquad 0 \leq \theta \leq \frac{1 - 2\sqrt{\gamma b} - \gamma b}{1 + 2\sqrt{\gamma b} + \gamma b}.$$

Then, the sequence $\{x_k\}$ generated by the inexact Newton method for solving $F(x) \in C$ with starting point x_0 and residual relative error tolerance θ: $x_{k+1} = x_k + d_k$,

$$d_k \in \operatorname{argmin}\left\{\|d\| \ : \ d \in \mathbb{X}, \ F(x_k) + F'(x_k)d + r_k \in C\right\}, \qquad \max_{w\in\{-r_k,\ r_k\}}\left\|T_{x_0}^{-1}w\right\| \leq \theta\left\|T_{x_0}^{-1}[-F(z_k)]\right\|,$$

for all $k = 0, 1, ...,$ *is well defined, for any particular choice of each* d_k,

$$\|T_{x_0}^{-1}[-F(x_k)]\| \leq \left(\frac{1+\theta^2}{2}\right)^k b, \qquad k = 0, 1, \dots ,$$

$\{x_k\}$ *is contained in* $B(x_0, \lambda)$ *and converges to a point* $x_* \in B[x_0, \lambda]$ *such that* $F(x_*) \in C$, *where*

$$\lambda := \frac{b}{\sqrt{\gamma b} + \gamma b}.$$

Moreover, letting $f : [0, 1/\gamma) \to \mathbb{R}$ *be defined by* $f(t) = t/(1-\gamma t) - 2t + b$, *the sequence* $\{x_k\}$ *satisfies*

$$\|x_k - x_{k+1}\| \leq \frac{1+\theta}{1-\theta}\left[\frac{1+\theta}{2}\frac{D^- f'(\lambda)}{|f'(\lambda)|}\|x_{k-1} - x_k\| + \theta\,\frac{2+f'(\lambda)}{|f'(\lambda)|}\right]\|x_{k-1} - x_k\|, \qquad k = 0, 1, \dots .$$

If, additionally, $0 \leq \theta < \left(-2(2 - 2\sqrt{\gamma b} - \gamma b) + \sqrt{5\gamma^2 b^2 - 44\sqrt{\gamma b} + 20\gamma b\sqrt{\gamma b} - 2\gamma b + 21}\right)/(5 - 2\sqrt{\gamma b} - \gamma b)$ *then* $\{x_k\}$ *converges Q-linearly as follows*

$$\limsup_{k\to\infty} \frac{\|x_* - x_{k+1}\|}{\|x_* - x_k\|} \leq \frac{1+\theta}{1-\theta}\left[\frac{1+\theta}{2} + \frac{2\theta}{1 - 2\sqrt{\gamma b} - \gamma b}\right], \qquad k = 0, 1, \dots .$$

Proof. Take $\tilde{\theta} = (1 - 2\sqrt{\gamma b} - \gamma b)/(1 + 2\sqrt{\gamma b} + \gamma b)$. Use Lemma 22 to prove that $f : [0, 1/\gamma) \to \mathbb{R}$ defined by $f(t) = t/(1-\gamma t) - 2t + b$, is a majorant function for F in x_0, see [14]. Therefore, all results follow from Theorem 7, applied to this particular context. □

Remark 5. *In Theorem 23, if* $\theta = 0$ *and* $C = \{0\}$ *then we obtain, in the setting of Smale's* α*-theory, [12, Theorem 21] for the exact Newton method and [15, Theorem 6.1] for the inexact Newton method, respectively.*

5 Final remarks

In this paper we have established a semi-local convergence analysis for inexact Newton's method for cone inclusion problem under affine invariant majorant condition. Following the same idea of this paper, as future works, we propose to study the exact and inexact Newton's method to the problem

$$F(x) + C(x) \ni 0, \tag{49}$$

described, respectively, by

$$F(x_k) + F'(x_k)(x_{k+1} - x_k) + C(x_{k+1}) \ni 0 \qquad k = 0, 1, \dots$$

and

$$(F(x_k) + F'(x_k)(x_{k+1} - x_k) + C(x_{k+1})) \cap R_k(x_k, x_{k+1}) \neq \varnothing, \qquad k = 0, 1, \dots ,$$

where $C : \mathbb{X} \times \mathbb{X} \rightrightarrows \mathbb{Y}$ is now a set-value mapping and $R_k : \mathbb{X} \times \mathbb{X} \rightrightarrows \mathbb{Y}$ is a sequence of set-value mappings with closed graphs. The problem (49) is a generalization to problem (1), which is called generalized equations, and it has been the subject of many new research, see [3, 9, 10, 11]. Furthermore, it will be interesting to study these two above methods under a majorant condition.

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