A semi-smooth Newton method for projection equations and linear complementarity problems with respect to the second order cone *

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Abstract

In this paper a special semi-smooth equation associated to the second order cone is studied. It is shown that, under mild assumptions, the semi-smooth Newton method applied to this equation is well-defined and the generated sequence is globally and Q-linearly convergent to a solution. As an application, the obtained results are used to study the linear second order cone complementarity problem, with special emphasis on the particular case of positive definite matrices. Moreover, some computational experiments designed to investigate the practical viability of the method are presented.

Keywords: Semi-smooth system, conic programming, second order cone, semi-smooth Newton method.

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1 Introduction

In this paper we consider the following special semi-smooth equation in x associated to the closed and convex cone $\mathcal{K} \subseteq \mathbb{R}^n$:

$$\mathrm{P}_{\mathcal{K}}(\mathrm{x}) + \mathrm{T}\mathrm{x} = b, \tag{1}$$

where $b \in \mathbb{R}^n$ is a constant vector, T is an $n \times n$ constant nonsingular real matrix and $\mathrm{P}_{\mathcal{K}}(\mathrm{x})$ denotes the Euclidean metric projection of a vector x onto the cone $\mathcal{K}$. The equation (1) associated to the positive orthant $\mathbb{R}^n_{++}$, was first studied in [6], additional papers dealing with (1) and its variations include [3–5, 7–9, 12, 15, 17, 24, 25, 31].

The purpose of the present paper is to discuss the semi-smooth Newton method to solve equation (1) associated to the *second order cone*

$$\mathcal{K} := \left\{ \mathrm{x} := (x_1, \mathrm{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \ : \ \|\mathrm{x}_2\| \le x_1 \right\}. \tag{2}$$

It is shown that, under mild assumptions, the semi-smooth Newton method applied to this equation is well-defined and the generated sequence is globally and Q-linearly convergent to a solution. As

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an application, we use the obtained results to study the *linear second order cone complementarity problem (LSOCCP)* : Find $\mathrm{x} \in \mathbb{R}^n$ such that

$$\mathrm{x} \in \mathcal{K}, \qquad \mathrm{Mx} + \mathrm{q} \in \mathcal{K}, \qquad \langle \mathrm{Mx} + \mathrm{q}, \mathrm{x} \rangle = 0, \tag{3}$$

where $\mathrm{q} \in \mathbb{R}^n$ is a constant vector, M is an $n \times n$ constant nonsingular real matrix. Complementarity problems related to the second order cone are considered in [14, 20, 23]. This topic of high interst is connected to several problems and has a wide range of applications, see [21]. Since this latter survey of applications many other important connections with physics, mechanics, economics, game theory, robotics, optimization and neural networks have been found, such as the ones in [2, 10, 19, 22, 27, 33, 34]. If M is symmetric, then the LSOCCP (3) is the optimality condition of the *quadratic programming problem under a second order cone constraint*,

$$\begin{array}{ll} \text{Minimize} & \frac{1}{2}\mathrm{x}^\top \mathrm{Mx} + \mathrm{q}^\top \mathrm{x} + c \\ & \mathrm{x} \in \mathcal{K} \end{array} \tag{4}$$

where c is any real number. Although not considered in this paper, it can be shown that any second order (in particular quadratic) conic optimization problem can be reformulated in terms of complementarity problems (in particular linear) related to the second order cone.

We show that our semi-smooth Newton method approach for solving problems (1), (3) and (4) has interesting features, for instance, the global and linear convergence of the generated sequence. Moreover, we present some computational experiments designed to investigate its practical viability. For a given class of problem, our numerical results suggest that the number of required iterations is almost unchanged. The numerical results also indicate a remarkable robustness with respect to the starting point.

The organization of the paper is as follows. In Section 1.1 some notations and auxiliary results used in the paper, are presented. In particular, important and useful properties of the projection mapping onto the second order cone are studied. In Section 2, we study the convergence properties of the semi-smooth Newton method for solving (1). In Section 3, the results of Section 2 are applied to find a solution of (3). In Section 4, we present some computational experiments. Final remarks are considered in Section 5.

1.1 Notations and preliminaries

In this section we present the notations and some auxiliary results used throughout the paper. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the canonical inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. If $\alpha \in \mathbb{R}$, then denote $\alpha^+ := \max\{\alpha, 0\}$ and $\alpha^- := \max\{-\alpha, 0\}$. The set of all $m \times n$ matrices with real entries is denoted by $\mathbb{R}^{m\times n}$ and $\mathbb{R}^n \equiv \mathbb{R}^{n\times 1}$. The matrix $\mathrm{Id_n}$ denotes the $n \times n$ identity matrix. Denote $\|\mathrm{E}\| := \max\{\|\mathrm{Ex}\| \ : \ \mathrm{x} \in \mathbb{R}^n, \ \|\mathrm{x}\| = 1\}$ for any $\mathrm{E} \in \mathbb{R}^{n\times n}$.

The next useful result, known as Banach's Lemma, which was proved in 2.1.1, page 32 of [29].

Lemma 1 (Banach's Lemma)**.** *Let* $\mathrm{E} \in \mathbb{R}^{n\times n}$*. If* $\|\mathrm{E}\| < 1$*, then* $\mathrm{E} - \mathrm{Id_n}$ *is invertible and* $\|(\mathrm{E} - \mathrm{Id_n})^{-1}\| \leq 1/(1 - \|\mathrm{E}\|)$.

We continue this section to present an interesting result on the eigenvalues of the sum of two symmetric matrices and its consequence.

Lemma 2. *Let A and B be two $n \times n$ symmetric matrices. Denote the eigenvalues of A, B and $A+B$ by $\lambda_i(A)$, $\lambda_i(B)$ and $\lambda_i(A+B)$ respectively, where $i = 1, \ldots, n$ and all three sets are arranged in non-increasing order. Then $\lambda_i(A) + \lambda_n(B) \leq \lambda_i(A+B) \leq \lambda_i(A) + \lambda_1(B)$ for $i = 1, \ldots, n$.*

Proof. See page 101 of [32]. $\square$

Corollary 1. *Let $A_1, \ldots, A_p$ be $n \times n$ symmetric matrices and $A = a_1 A_1 + \cdots + a_p A_p$ such that $a_1 + \ldots + a_p = 1$ with $a_j \in [0, 1]$. Denote the eigenvalues of A_j and A by $\lambda_i(A_j)$ and $\lambda_i(A)$ respectively, where $j = 1, \ldots, p$, $i = 1, \ldots, n$ and all the sets are arranged in non-increasing order. Then $\min\{\lambda_n(A_1), \ldots, \lambda_n(A_p)\} \leq \lambda_i(A) \leq \max\{\lambda_1(A_1), \ldots, \lambda_1(A_p)\}$, for $i = 1, \ldots, n$.*

Proof. Since $a_1 + \ldots + a_p = 1$ with $a_j \in [0, 1]$ and the eigenvalues of $a_j A_j$ are equals to $a_j \lambda_i(A_j)$ for $j = 1, \ldots, p$ and $i = 1, \ldots, n$. Thus, the proof is an immediate consequence of Lemma 2. $\square$

For a differentiable mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ at a point $\mathrm{x} \in \mathbb{R}^n$, we denote by $F'(\mathrm{x}) \in \mathbb{R}^{m \times n}$ the Jacobian of F at this point. If F is a locally Lipschitz continuous mapping, then the set

$$\partial_B F(\mathrm{x}) := \left\{ \mathrm{V}(\mathrm{x}) \in \mathbb{R}^{m \times n} \ : \ \exists \ \{\mathrm{x}^k\} \subset D_F; \ \mathrm{x}^k \to \mathrm{x}, \ F'(\mathrm{x}^k) \to \mathrm{V}(\mathrm{x}) \right\},$$

is nonempty and called the *B-subdifferential* of F at x, where $D_F \subseteq \mathbb{R}^n$ denotes the set of points at which F is differentiable. The convex hull of $\partial_B F$, $\partial F(\mathrm{x}) := \mathrm{conv}\, \partial_B F(\mathrm{x})$, is known as the *Clarke's generalized Jacobian*, see Definition 2.6.1 on page 70 of [11]. The next result is the generalization of the vector *Mean-Value Theorem* for Lipschitz continuous mapping, see Proposition 2.6.5 on page 72 of [11].

Proposition 1. *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz continuous mapping. Then we have*

$$F(\mathrm{y}) - F(\mathrm{x}) \in \mathrm{conv}\, \partial F([\mathrm{x}, \mathrm{y}])(\mathrm{y} - \mathrm{x}), \qquad \mathrm{x}, \mathrm{y} \in \mathbb{R}^n,$$

where the right hand side in the inclusion denotes the convex hull of all points of the form $U(\mathrm{z})(\mathrm{y}-\mathrm{x})$ with $U(\mathrm{z}) \in \partial F(\mathrm{z})$ and $\mathrm{z} \in [\mathrm{x}, \mathrm{y}] := \{t\mathrm{x} + (1-t)\mathrm{y} \ : \ t \in [0, 1]\}$.

We end this section with the well-known contraction mapping principle, see 8.2.2, page 153 of [29].

Theorem 1 (Contraction mapping principle)**.** *Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$. Suppose that there exists $\kappa \in [0, 1)$ such that $\|\Phi(\mathrm{y}) - \Phi(\mathrm{x})\| \leq \kappa \|\mathrm{y} - \mathrm{x}\|$, for all $\mathrm{x}, \mathrm{y} \in \mathbb{R}^n$. Then there exists a unique $\bar{\mathrm{x}} \in \mathbb{R}^n$ such that $\Phi(\bar{\mathrm{x}}) = \bar{\mathrm{x}}$.*

1.2 Properties of the projetion mapping

In this section we present some properties of the projection mapping onto a second order cone, which will play important roles in the study of equation in (1). We begin with some notations.

The *polar cone* and the *dual cone* of second order cone $\mathcal{K}$ in (2) are, respectively, the sets

$$\mathcal{K}^{\circ} := \{\mathrm{x} \in \mathbb{R}^n \ : \ \langle \mathrm{x}, \mathrm{y} \rangle \leq 0, \ \forall\, \mathrm{y} \in \mathcal{K}\}, \qquad \mathcal{K}^{*} := \{\mathrm{x} \in \mathbb{R}^n \ : \ \langle \mathrm{x}, \mathrm{y} \rangle \geq 0, \ \forall\, \mathrm{y} \in \mathcal{K}\}.$$

Recall that $\mathcal{K}^{\circ} = -\mathcal{K}^{*}$ and $\mathcal{K}$ is self-dual, i.e., $\mathcal{K}^{*} = \mathcal{K}$. Moreover, it follows from Moreau's decomposition theorem, see [26] (see also [16, Theorem 3.2.5]), that

$$\mathrm{x} = \mathrm{P}_{\mathcal{K}}(\mathrm{x}) - \mathrm{P}_{\mathcal{K}}(-\mathrm{x}), \qquad \langle \mathrm{P}_{\mathcal{K}}(\mathrm{x}), \ \mathrm{P}_{\mathcal{K}}(-\mathrm{x}) \rangle = 0, \qquad \mathrm{x} \in \mathbb{R}^n. \tag{5}$$

An explicit representation of the projection mapping $\mathrm{P}_{\mathcal{K}}$ onto $\mathcal{K}$ is given in the following result, see [13, Proposition 3.3].

Lemma 3. *Let* $\mathrm{x} := (x_1, \mathrm{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ *and* $\mathcal{K}$ *be the second order cone. Then,*

$$\mathrm{P}_{\mathcal{K}}(\mathrm{x}) = \begin{cases} \dfrac{1}{2}\left((x_1 - \|\mathrm{x}_2\|)^+ + (x_1 + \|\mathrm{x}_2\|)^+,\ \left[(x_1 + \|\mathrm{x}_2\|)^+ - (x_1 - \|\mathrm{x}_2\|)^+\right] \dfrac{\mathrm{x}_2}{\|\mathrm{x}_2\|} \right), & \mathrm{x}_2 \neq 0, \\ \\ \left(x_1^+,\, 0\right), & \mathrm{x}_2 = 0. \end{cases} \tag{6}$$

Remark 1. *It is well-know that the orthogonal projection onto a closed convex set is continuous and firmly nonexpansive. In particular,* $\|\mathrm{P}_{\mathcal{K}}(\mathrm{x}) - \mathrm{P}_{\mathcal{K}}(\mathrm{y})\| \leq \|\mathrm{x} - \mathrm{y}\|$ *for all* $\mathrm{x}, \mathrm{y} \in \mathbb{R}^n$*, see [16].*

Since $\mathrm{P}_{\mathcal{K}}$ is nonexpansive, it is Lipschitz continuous. Thus, by *Rademacher's Theorem*, we conclude that $\mathrm{P}_{\mathcal{K}}$ is differentiable everywhere except at $x_1 = \pm\|\mathrm{x}_2\|$. In the next result we present the Jacobian of $\mathrm{P}_{\mathcal{K}}$ at a point where it is differentiable, and as a consequence, we derive some important properties of its B-subdifferential.

Lemma 4. *The projection mapping* $\mathrm{P}_{\mathcal{K}}$ *onto the second order cone* $\mathcal{K}$ *is continuously differentiable at every* $\mathrm{x} := (x_1, \mathrm{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ *such that* $x_1 \neq \pm\|\mathrm{x}_2\|$ *and its Jacobian is given by*

$$\mathrm{P}'_{\mathcal{K}}(\mathrm{x}) := \begin{cases} \mathrm{Id_n} & x_1 > \|\mathrm{x}_2\|, \\ 0 & x_1 < -\|\mathrm{x}_2\|, \\ \dfrac{1}{2}\begin{bmatrix} 1 & \mathrm{w}^\top \\ \mathrm{w} & \mathrm{H} \end{bmatrix} & -\|\mathrm{x}_2\| < x_1 < \|\mathrm{x}_2\|, \end{cases} \tag{7}$$

where $\mathrm{w} = \mathrm{x}_2/\|\mathrm{x}_2\|$ *and* $H = [1 + x_1/\|\mathrm{x}_2\|]\,\mathrm{Id_{n-1}} - (x_1/\|\mathrm{x}_2\|)\mathrm{w}\mathrm{w}^\top$*. As a consequence, at each* $\mathrm{x} \in \mathbb{R}^n$*, the matrix* $\mathrm{V}(\mathrm{x}) \in \partial_B \mathrm{P}_{\mathcal{K}}(\mathrm{x})$ *has the following representation:*

(a) *If* $x_1 \neq \pm\|\mathrm{x}_2\|$*, then* $\mathrm{V}(\mathrm{x}) = \mathrm{P}'_{\mathcal{K}}(\mathrm{x})$*;*

(b) *If* $\mathrm{x}_2 \neq 0$ *and* $x_1 = \|\mathrm{x}_2\|$*, then* $\mathrm{V}(\mathrm{x}) = \mathrm{Id_n}$ *or*

$$\mathrm{V}(\mathrm{x}) = \frac{1}{2}\begin{bmatrix} 1 & \mathrm{w}^\top \\ \mathrm{w} & \mathrm{H} \end{bmatrix}, \qquad \mathrm{w} = \frac{\mathrm{x}_2}{\|\mathrm{x}_2\|}, \qquad \mathrm{H} = 2\mathrm{Id_{n-1}} - \mathrm{w}\mathrm{w}^\top; \tag{8}$$

(c) *If* $\mathrm{x}_2 \neq 0$ *and* $x_1 = -\|\mathrm{x}_2\|$*, then* $\mathrm{V}(\mathrm{x}) = 0$ *or*

$$\mathrm{V}(\mathrm{x}) = \frac{1}{2}\begin{bmatrix} 1 & \mathrm{w}^\top \\ \mathrm{w} & \mathrm{H} \end{bmatrix}, \qquad \mathrm{w} = \frac{\mathrm{x}_2}{\|\mathrm{x}_2\|}, \qquad \mathrm{H} = \mathrm{w}\mathrm{w}^\top; \tag{9}$$

(d) *If* $\mathrm{x}_2 = 0$ *and* $x_1 = 0$*, then* $\mathrm{V}(\mathrm{x}) = 0$ *or* $\mathrm{V}(\mathrm{x}) = \mathrm{Id_n}$ *or* $\mathrm{V}(\mathrm{x})$ *belongs to the set*

$$\left\{ \frac{1}{2}\begin{bmatrix} 1 & \mathrm{w}^\top \\ \mathrm{w} & \mathrm{H} \end{bmatrix} \ :\ \mathrm{H} = (1+\rho)\mathrm{Id_{n-1}} - \rho\mathrm{w}\mathrm{w}^\top,\ \textit{for some } |\rho| < 1 \textit{ and } \|\mathrm{w}\| = 1 \right\}.$$

Moreover, the eigenvalues of any matrix $\mathrm{V}(\mathrm{x}) \in \partial_B \mathrm{P}_{\mathcal{K}}(\mathrm{x})$ *belong to the interval* $[0, 1]$*, and consequently* $\|\mathrm{V}(\mathrm{x})\| \leq 1$*, for all* $\mathrm{x} \in \mathbb{R}^n$*.*

Proof. Combine Lemmas 2.5, 2.6 and 2.8 of [18]. □

In the next two lemmas, we obtain important properties of the orthogonal projection $\mathrm{P}_{\mathcal{K}}$, which will be used in the definition and the convergence analysis of the semi-smooth Newton method for solving equation in (1).

Lemma 5. *For every* $\mathrm{x} := (x_1, \mathrm{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ *and* $\mathrm{V}(\mathrm{x}) \in \partial_B \mathrm{P}_{\mathcal{K}}(\mathrm{x})$ *there holds* $\mathrm{V}(\mathrm{x})\mathrm{x} = \mathrm{P}_{\mathcal{K}}(\mathrm{x})$.

Proof. Take any $\mathrm{x} = (x_1, \mathrm{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$. Then, the analysis will be done following the four possibilities from **(a)** to **(d)** as in the statement of Lemma 4. First we assume **(a)**, that is $x_1 \neq \pm\|\mathrm{x}_2\|$. Then, $\mathrm{V}(\mathrm{x}) = \mathrm{P}'_{\mathcal{K}}(\mathrm{x})$. In this case, by using (7), we conclude that

$$\mathrm{P}'_{\mathcal{K}}(\mathrm{x})\mathrm{x} = \begin{cases} \mathrm{x} & x_1 > \|\mathrm{x}_2\|, \\ 0 & x_1 < -\|\mathrm{x}_2\|, \\ \dfrac{1}{2}\left(x_1 + \|\mathrm{x}_2\|,\ (x_1 + \|\mathrm{x}_2\|)\dfrac{\mathrm{x}_2}{\|\mathrm{x}_2\|}\right) & -\|\mathrm{x}_2\| < x_1 < \|\mathrm{x}_2\|. \end{cases}$$

On the other hand, by using (6), we easily conclude that $\mathrm{P}'_{\mathcal{K}}(\mathrm{x})\mathrm{x} = \mathrm{P}_{\mathcal{K}}(\mathrm{x})$.

Next assume the conditions of **(b)**, that is, $\mathrm{x}_2 \neq 0$ and $x_1 = \|\mathrm{x}_2\|$. If $\mathrm{V}(\mathrm{x}) = \mathrm{Id_n}$, then $\mathrm{V}(\mathrm{x})\mathrm{x} = \mathrm{x}$. If $\mathrm{V}(\mathrm{x})$ is given by (8), then

$$\mathrm{V}(\mathrm{x})\mathrm{x} = \frac{1}{2}\left(x_1 + \|\mathrm{x}_2\|,\ (x_1 + \|\mathrm{x}_2\|)\frac{\mathrm{x}_2}{\|x_2\|}\right),$$

and using the assumption $x_1 = \|\mathrm{x}_2\|$ we conclude $\mathrm{V}(\mathrm{x})\mathrm{x} = \mathrm{x}$. Now, taking into account that $\mathrm{x}_2 \neq 0$, $x_1 - \|\mathrm{x}_2\| = 0$ and $x_1 + \|\mathrm{x}_2\| > 0$ the equality in (6) becomes $\mathrm{P}_{\mathcal{K}}(\mathrm{x}) = \mathrm{x}$. Therefore, we also have $\mathrm{V}(\mathrm{x}) = \mathrm{P}_{\mathcal{K}}(\mathrm{x})$.

Now we assume the conditions of **(c)**, that is $\mathrm{x}_2 \neq 0$ and $x_1 = -\|\mathrm{x}_2\|$. If $\mathrm{V}(\mathrm{x}) = 0$ then $\mathrm{V}(\mathrm{x})\mathrm{x} = 0$. If $\mathrm{V}(\mathrm{x})$ is given by (9) then we have

$$\mathrm{V}(\mathrm{x})\mathrm{x} = \frac{1}{2}\left(x_1 + \|\mathrm{x}_2\|,\ (x_1 + \|\mathrm{x}_2\|)\frac{\mathrm{x}_2}{\|x_2\|}\right),$$

and because $x_1 + \|\mathrm{x}_2\| = 0$, we also conclude that $\mathrm{V}(\mathrm{x})\mathrm{x} = 0$. Note that the assumptions $\mathrm{x}_2 \neq 0$ and $x_1 = -\|\mathrm{x}_2\|$ imply $x_1 + \|\mathrm{x}_2\| = 0$ and $x_1 - \|\mathrm{x}_2\| < 0$. Thus, the equality in (6) implies $\mathrm{P}_{\mathcal{K}}(\mathrm{x}) = 0$. Therefore, we also have $\mathrm{V}(\mathrm{x}) = \mathrm{P}_{\mathcal{K}}(\mathrm{x})$.

Finally, we prove the statement of **(d)**, assuming that $\mathrm{x}_2 = 0$ and $x_1 = 0$. It follows trivially that $\mathrm{V}(\mathrm{x})\mathrm{x} = 0$ and $\mathrm{P}_{\mathcal{K}}(\mathrm{x}) = 0$. Thus, we also have $\mathrm{V}(\mathrm{x})\mathrm{x} = \mathrm{P}_{\mathcal{K}}(\mathrm{x})$.

Therefore, we conclude that in all possible cases the lemma holds for every $\mathrm{x} = (x_1, \mathrm{x}_2) \in \mathbb{R}\times\mathbb{R}^{n-1}$, which concludes the proof. □

Lemma 6. *Let* $\mathrm{x}, \mathrm{y} \in \mathbb{R}^n$ *and* $\mathrm{V}(\mathrm{x}) \in \partial_B \mathrm{P}_{\mathcal{K}}(\mathrm{x})$. *Then* $\|\mathrm{P}_{\mathcal{K}}(\mathrm{y}) - \mathrm{P}_{\mathcal{K}}(\mathrm{x}) - \mathrm{V}(\mathrm{x})(\mathrm{y} - \mathrm{x})\| \leq \|\mathrm{y} - \mathrm{x}\|$.

Proof. After simple algebraic manipulation we conclude from Proposition 1 that

$$\mathrm{P}_{\mathcal{K}}(\mathrm{y}) - \mathrm{P}_{\mathcal{K}}(\mathrm{x}) - \mathrm{V}(\mathrm{x})(\mathrm{y} - \mathrm{x}) \in \operatorname{conv} \partial \mathrm{P}_{\mathcal{K}}([\mathrm{x}, \mathrm{y}])(\mathrm{y} - \mathrm{x}) - \mathrm{V}(\mathrm{x})(\mathrm{y} - \mathrm{x}), \qquad \mathrm{x}, \mathrm{y} \in \mathbb{R}^n. \tag{10}$$

On the other hand, Lemma 4 implies that the eigenvalues of any matrix $\mathrm{V}(\mathrm{x}) \in \partial_B \mathrm{P}_{\mathcal{K}}(\mathrm{x})$ belong to the interval $[0, 1]$, for all $\mathrm{x} \in \mathbb{R}^n$. Thus, by combining the definitions of $\partial \mathrm{P}_{\mathcal{K}}$ and $\operatorname{conv} \partial \mathrm{P}_{\mathcal{K}}([\mathrm{x}, \mathrm{y}])$ with Corollary 1, we conclude that the eigenvalues of any matrix $\mathrm{U} \in \operatorname{conv} \partial \mathrm{P}_{\mathcal{K}}([\mathrm{x}, \mathrm{y}])$ also belong to $[0, 1]$. Therefore, letting $\mathrm{U} \in \operatorname{conv} \partial \mathrm{P}_{\mathcal{K}}([\mathrm{x}, \mathrm{y}])$ and $\mathrm{V}(\mathrm{x}) \in \partial_B \mathrm{P}_{\mathcal{K}}(\mathrm{x})$ and taking into account that the eigenvalues of U and $\mathrm{V}(\mathrm{x})$ belong to $[0, 1]$, we conclude from Lemma 2 that the eigenvalues of $\mathrm{U} - \mathrm{V}(\mathrm{x})$ belong to the interval $[-1, 1]$. Hence, since $\mathrm{U} - \mathrm{V}(\mathrm{x})$ is a symmetric matrix, we have $\|\mathrm{U} - \mathrm{V}(\mathrm{x})\| \leq 1$. On the other hand, (10) implies that there exists $\mathrm{U} \in \operatorname{conv} \partial \mathrm{P}_{\mathcal{K}}([\mathrm{x}, \mathrm{y}])$ such that

$$\mathrm{P}_{\mathcal{K}}(\mathrm{y}) - \mathrm{P}_{\mathcal{K}}(\mathrm{x}) - \mathrm{V}(\mathrm{x})(\mathrm{y} - \mathrm{x}) = (\mathrm{U} - \mathrm{V}(\mathrm{x}))(\mathrm{y} - \mathrm{x}).$$

By taking the norm in the above equality and by using that $\|\mathrm{U} - \mathrm{V}(\mathrm{x})\| \leq 1$, we get that the desired inequality holds. □

2 A semi-smooth Newton method

In this section, we present and analyze the semi-smooth Newton method for solving (1). We begin presenting an existence result of solution of equation (1).

Proposition 2. *If* $\|\mathrm{T}^{-1}\| < 1$ *then* (1) *has a unique solution for any* $b \in \mathbb{R}^n$*.*

Proof. The equation (1) has a solution if only if $\Phi(\mathrm{x}) = -\mathrm{T}^{-1}\mathrm{P}_{\mathcal{K}}(\mathrm{x}) + \mathrm{T}^{-1}b$ has a fixed point. It follows from definition of Φ that

$$\Phi(\mathrm{x}) - \Phi(\mathrm{y}) = -\mathrm{T}^{-1}(\mathrm{P}_{\mathcal{K}}(\mathrm{x}) - \mathrm{P}_{\mathcal{K}}(\mathrm{y})), \qquad \mathrm{x}, \mathrm{y} \in \mathbb{R}^n.$$

Since $\|\mathrm{T}^{-1}\| < 1$ and $\|\mathrm{P}_{\mathcal{K}}(\mathrm{x}) - \mathrm{P}_{\mathcal{K}}(\mathrm{y})\| \leq \|\mathrm{x} - \mathrm{y}\|$ for all $\mathrm{x}, \mathrm{y} \in \mathbb{R}^n$ and after taking the norm in the last equality, we get $\|\Phi(\mathrm{x}) - \Phi(\mathrm{y})\| \leq \|\mathrm{T}^{-1}\| \|\mathrm{x} - \mathrm{y}\|$, for all $\mathrm{x}, \mathrm{y} \in \mathbb{R}^n$. Hence Φ is a contraction. Therefore, by applying Theorem 1, we conclude that Φ has precisely an unique fixed point and consequently (1) has a unique solution. □

The next example shows that the bound $\|\mathrm{T}^{-1}\| < 1$ in Proposition 2 is strict.

Example 1. *Consider equation* (1) *where*

$$\mathrm{T} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \textit{and} \qquad \mathrm{b} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Note that $\|\mathrm{T}^{-1}\| = 1$*. By using* (6)*, direct calculations show that* $[1 \quad 1]^\top$ *and* $[1 \quad -1]^\top$ *are solutions of* (1)*.*

The *semi-smooth Newton method*, introduced in [30], for finding the zero of the semi-smooth function

$$F(\mathrm{x}) := \mathrm{P}_{\mathcal{K}}(\mathrm{x}) + \mathrm{T}\mathrm{x} - b, \qquad \mathrm{x} \in \mathbb{R}^n, \tag{11}$$

with starting point $\mathrm{x}^0 \in \mathbb{R}^n$, it is formally defined by

$$F(\mathrm{x}^k) + \mathrm{U}(\mathrm{x}^k)\left(\mathrm{x}^{k+1} - \mathrm{x}^k\right) = 0, \qquad \mathrm{U}(\mathrm{x}^k) \in \partial F(\mathrm{x}^k), \qquad k = 0, 1, \ldots. \tag{12}$$

Note that $\mathrm{U}(\mathrm{x}^k)$ is any subgradient in $\partial F(\mathrm{x}^k)$, the Clarke generalized Jacobian of F at x^k. By letting

$$\mathrm{V}(\mathrm{x}) \in \partial_B \mathrm{P}_{\mathcal{K}}(\mathrm{x}), \qquad \mathrm{x} \in \mathbb{R}^n, \tag{13}$$

it easy to see from (11) that $\mathrm{V}(\mathrm{x}) + \mathrm{T} \in \partial F(\mathrm{x})$. Since Lemma 5 implies that $\mathrm{V}(\mathrm{x})\mathrm{x} = \mathrm{P}_{\mathcal{K}}(\mathrm{x})$ for all $\mathrm{x} \in \mathbb{R}^n$ and $\mathrm{V}(\mathrm{x}) \in \partial_B \mathrm{P}_{\mathcal{K}}(\mathrm{x})$, by taking $\mathrm{U}(\mathrm{x}^k) = \mathrm{V}(\mathrm{x}^k) + \mathrm{T}$, equation (12) becomes

$$\left[\mathrm{V}(\mathrm{x}^k) + \mathrm{T}\right]\mathrm{x}^{k+1} = b, \qquad \mathrm{V}(\mathrm{x}^k) \in \partial_B \mathrm{P}_{\mathcal{K}}(\mathrm{x}^k), \qquad k = 0, 1, \ldots, \tag{14}$$

which formally defines the *semi-smooth Newton sequence* $\{\mathrm{x}^k\}$ for solving (1). It is worth mentioning that a similar iteration was studied in [6].

The next proposition gives a stopping condition for the semi-smooth Newton iteration given in (14).

Proposition 3. *If in* (14) $\mathrm{V}(\mathrm{x}^{k+1}) = \mathrm{V}(\mathrm{x}^k)$*, then* x^{k+1} *is a solution of* (1)*.*

Proof. Since $\mathrm{V}(\mathrm{x}^{k+1}) = \mathrm{V}(\mathrm{x}^k)$, the equation in (14) gives us $\left[\mathrm{V}(\mathrm{x}^{k+1}) + \mathrm{T}\right] \mathrm{x}^{k+1} = b$. The equation $\mathrm{V}(\mathrm{x}^{k+1})\mathrm{x}^{k+1} = \mathrm{P}_{\mathcal{K}}(\mathrm{x}^{k+1})$, together with the previous one, yield $\mathrm{P}_{\mathcal{K}}(\mathrm{x}^{k+1}) + \mathrm{T}\mathrm{x}^{k+1} = b$, which implies that x^{k+1} is a solution of (1). □

The sufficient condition for the Q-linear convergence of the sequence generated by (14) is presented in the next theorem.

Theorem 2. *Let $b \in \mathbb{R}^n$ and $\mathrm{T} \in \mathbb{R}^{n\times n}$ be a nonsingular matrix. Assume that $\left\|\mathrm{T}^{-1}\right\| < 1$. Then, (1) has a unique solution $\mathrm{x}^* \in \mathbb{R}^n$ and, for any starting point $\mathrm{x}^0 \in \mathbb{R}^n$, the semi-smooth Newton sequence $\{\mathrm{x}^k\}$ generated by (14) is well-defined. Additionally, if*

$$\left\|\mathrm{T}^{-1}\right\| < 1/2, \tag{15}$$

then the sequence $\{\mathrm{x}^k\}$ converges Q-linearly to x^* as follows:

$$\|\mathrm{x}^* - \mathrm{x}^{k+1}\| \leq \frac{\|\mathrm{T}^{-1}\|}{1 - \|\mathrm{T}^{-1}\|}\|\mathrm{x}^* - \mathrm{x}^k\|, \qquad k = 0, 1, \ldots. \tag{16}$$

Proof. Let $\mathrm{x} \in \mathbb{R}^n$. It follows from Lemma 4 that $\|\mathrm{V}(\mathrm{x})\| \leq 1$ for all $\mathrm{x} \in \mathbb{R}^n$. Hence, by using the properties of the norm and by taking into account that $\left\|\mathrm{T}^{-1}\right\| < 1$, we conclude that $\|\mathrm{T}^{-1}\mathrm{V}(\mathrm{x})\| < 1$. Thus, Lemma 1 implies that $-\mathrm{T}^{-1}\mathrm{V}(\mathrm{x}) - \mathrm{Id_n}$ is nonsingular. Since T is nonsingular and

$$\mathrm{V}(\mathrm{x}) + \mathrm{T} = -\mathrm{T}\left[-\mathrm{T}^{-1}\mathrm{V}(\mathrm{x}) - \mathrm{Id_n}\right], \qquad x \in \mathbb{R}^n,$$

we obtain that $\mathrm{V}(\mathrm{x}) + \mathrm{T}$ is also nonsingular. Therefore, for any starting point $\mathrm{x}^0 \in \mathbb{R}^n$, (14) implies that the sequence $\{\mathrm{x}^k\}$ generated by (14) is well-defined.

By using Proposition 2, we conclude that (1) has a unique solution $\mathrm{x}^* \in \mathbb{R}^n$. Since $\mathrm{x}^* \in \mathbb{R}^n$ is the solution of (1), we have $[\mathrm{V}(\mathrm{x}^*) + \mathrm{T}]\mathrm{x}^* - b = 0$, which together with the definition of $\{\mathrm{x}^k\}$ in (14) and (13), implies

$$\begin{aligned}\mathrm{x}^* - \mathrm{x}^{k+1} &= -[\mathrm{V}(\mathrm{x}^k) + \mathrm{T}]^{-1}\left[b - [\mathrm{V}(\mathrm{x}^k) + \mathrm{T}]\mathrm{x}^*\right] \\ &= -[\mathrm{V}(\mathrm{x}^k) + \mathrm{T}]^{-1}\left[[\mathrm{V}(\mathrm{x}^*) + \mathrm{T}]\mathrm{x}^* - b - [\mathrm{V}(\mathrm{x}^k) + \mathrm{T}]\mathrm{x}^k + b - [\mathrm{V}(\mathrm{x}^k) + \mathrm{T}](\mathrm{x}^* - \mathrm{x}^k)\right],\end{aligned}$$

for $k = 0, 1, \ldots$. On the other hand, since $\mathrm{V}(\mathrm{x})\mathrm{x} = \mathrm{P}_{\mathcal{K}}(\mathrm{x})$ for all $\mathrm{x} \in \mathbb{R}^n$, after simple algebraic manipulations we obtain

$$[\mathrm{V}(\mathrm{x}^*) + \mathrm{T}]\mathrm{x}^* - b - [\mathrm{V}(\mathrm{x}^k) + \mathrm{T}]\mathrm{x}^k + b - [\mathrm{V}(\mathrm{x}^k) + \mathrm{T}](\mathrm{x}^* - \mathrm{x}^k) = \mathrm{P}_{\mathcal{K}}(\mathrm{x}^*) - \mathrm{P}_{\mathcal{K}}(\mathrm{x}^k) - \mathrm{V}(\mathrm{x}^k)(\mathrm{x}^* - \mathrm{x}^k),$$

for $k = 0, 1, \ldots$. By combining the above two equalities and by using the properties of the norm, we obtain

$$\|\mathrm{x}^* - \mathrm{x}^{k+1}\| \leq \left\|[\mathrm{V}(\mathrm{x}^k) + \mathrm{T}]^{-1}\right\| \left\|\mathrm{P}_{\mathcal{K}}(\mathrm{x}^*) - \mathrm{P}_{\mathcal{K}}(\mathrm{x}^k) - \mathrm{V}(\mathrm{x}^k)(\mathrm{x}^* - \mathrm{x}^k)\right\|, \qquad k = 0, 1, \ldots. \tag{17}$$

It follows from Lemma 6 that $\left\|\mathrm{P}_{\mathcal{K}}(\mathrm{x}^*) - \mathrm{P}_{\mathcal{K}}(\mathrm{x}^k) - \mathrm{V}(\mathrm{x}^k)(\mathrm{x}^* - \mathrm{x}^k)\right\| \leq \|\mathrm{x}^* - \mathrm{x}^k\|$, for $k = 0, 1, \ldots$, and, by combining the last inequality with (17), we get

$$\|\mathrm{x}^* - \mathrm{x}^{k+1}\| \leq \left\|[\mathrm{V}(\mathrm{x}^k) + \mathrm{T}]^{-1}\right\| \|\mathrm{x}^* - \mathrm{x}^k\|, \qquad k = 0, 1, \ldots. \tag{18}$$

On the other hand, by using the properties of the norm, after some simple algebraic manipulations, we get

$$\left\|[\mathrm{V}(\mathrm{x}^k) + \mathrm{T}]^{-1}\right\| = \left\|[-\mathrm{T}^{-1}\mathrm{V}(\mathrm{x}^k) - \mathrm{Id_n}]^{-1}\left[-\mathrm{T}^{-1}\right]\right\| \leq \left\|[\mathrm{T}^{-1}\mathrm{V}(\mathrm{x}^k) + \mathrm{Id_n}]^{-1}\right\| \|\mathrm{T}^{-1}\|, \quad k = 0, 1, \ldots,$$

which combined with Lemma 1 and $\|\mathrm{T}^{-1}\mathrm{V}(\mathrm{x}^k)\| \leq \|\mathrm{T}^{-1}\| < 1$ implies

$$\left\|[\mathrm{V}(\mathrm{x}^k)+\mathrm{T}]^{-1}\right\| \leq \frac{\|\mathrm{T}^{-1}\|}{1-\|\mathrm{T}^{-1}\|}, \qquad k=0,1,\ldots.$$

Thus, the last inequality and (18) yield (16). Note that (15) implies $\|\mathrm{T}^{-1}\|/(1-\|\mathrm{T}^{-1}\|) < 1$. Therefore, (16) implies that $\{\mathrm{x}^k\}$ converges Q-linearly, from any starting point x^0, to the solution x^* of (1). Hence, the theorem is proven. □

If T is a symmetric and positive definite matrix, then stronger results are obtained.

Theorem 3. *Let $b \in \mathbb{R}^n$ and $\mathrm{T} \in \mathbb{R}^{n\times n}$ be a symmetric and positive definite matrix. Then* (1) *has a unique solution $\mathrm{x}^* \in \mathbb{R}^n$ and, for any starting point $\mathrm{x}^0 \in \mathbb{R}^n$, the semi-smooth Newton sequence $\{\mathrm{x}^k\}$ generated by* (14) *is well-defined. Moreover, if $\left\|\mathrm{T}^{-1}\right\| < 1$, then $\{\mathrm{x}^k\}$ converges Q-linearly to x^* as follows:*

$$\|\mathrm{x}^*-\mathrm{x}^{k+1}\| \leq \|\mathrm{T}^{-1}\|\|\mathrm{x}^*-\mathrm{x}^k\|, \qquad k=0,1,\ldots.$$

Proof. Since T is symmetric and positive definite, it follows from Lemma 2 that $\mathrm{Id_n}+\mathrm{T}$ is nonsingular. Thus, taking into account that $\mathrm{x} = \mathrm{P}_{\mathcal{K}}(\mathrm{x}) - \mathrm{P}_{\mathcal{K}}(-\mathrm{x})$, after some algebraic manipulations, we conclude that (1) is equivalent to $\mathrm{x} = [\mathrm{Id_n}+\mathrm{T}]^{-1}(b-\mathrm{P}_{\mathcal{K}}(-\mathrm{x}))$. Therefore, equation (1) has a solution if only if $\Phi(\mathrm{x}) = [\mathrm{Id_n}+\mathrm{T})]^{-1}(b-\mathrm{P}_{\mathcal{K}}(-\mathrm{x}))$ has a fixed point. On the other hand, it follows from the definition of Φ that

$$\Phi(\mathrm{x})-\Phi(\mathrm{y}) = [\mathrm{Id_n}+\mathrm{T}]^{-1}(-\mathrm{P}_{\mathcal{K}}(-\mathrm{x})+\mathrm{P}_{\mathcal{K}}(-\mathrm{y})), \qquad \mathrm{x},\mathrm{y}\in\mathbb{R}^n.$$

Since T is symmetric and positive definite, it follows from Lemma 2 that $\left\|[\mathrm{Id_n}+\mathrm{T}]^{-1}\right\| = \kappa < 1$, where $\kappa = 1/(1+\lambda_{\min})$ and $\lambda_{\min} > 0$ is the minimum eigenvalue of T. Now, proceeding as in Proposition 2, it is possible to conclude that Φ is a contraction and it has precisely a unique fixed point. Consequently (1) has a unique solution.

Lemma 4 implies that the eigenvalues of $\mathrm{V}(\mathrm{x})$ belongs to the interval $[0,1]$, for all $\mathrm{x}\in\mathbb{R}^n$. Hence, the nonsingularity of $\mathrm{V}(\mathrm{x})+\mathrm{T}$ follows from Lemma 2. As a consequence, the sequence $\{\mathrm{x}^k\}$ generated by (14) is well-defined for any starting point. In order to prove the Q-linear convergence of $\{\mathrm{x}^k\}$ to $\mathrm{x}^*\in\mathbb{R}^n$, the unique solution of (1), we proceed as in the proof of Theorem 2 to obtain

$$\|\mathrm{x}^*-\mathrm{x}^{k+1}\| \leq \left\|[\mathrm{V}(\mathrm{x}^k)+\mathrm{T}]^{-1}\right\|\|\mathrm{x}^*-\mathrm{x}^k\|, \qquad k=0,1,\ldots.$$

Lemma 2 allows us to conclude that $\left\|[\mathrm{V}(\mathrm{x}^k)+\mathrm{T}]^{-1}\right\| \leq \|\mathrm{T}^{-1}\|$. Thus, by combining the latter two inequalities we have

$$\|\mathrm{x}^*-\mathrm{x}^{k+1}\| \leq \|\mathrm{T}^{-1}\|\|\mathrm{x}^*-\mathrm{x}^k\|, \qquad k=0,1,\ldots.$$

Therefore, as we are under the assumption $\left\|\mathrm{T}^{-1}\right\| < 1$, the last inequality implies that $\{\mathrm{x}^k\}$ converges Q-linearly to $\mathrm{x}^*\in\mathbb{R}^n$, from any starting point x^0. □

The invertibility of $\mathrm{V}(\mathrm{x})+\mathrm{T}$, for all $\mathrm{x}\in\mathbb{R}^n$, is sufficient for the well-definedness of the Newton method. However, the next example shows that an additional condition on T must be assumed for convergence, for instance, (15).

Example 2. *Consider the function $F:\mathbb{R}^2\to\mathbb{R}^2$ defined by $F(\mathrm{x}) = \mathrm{P}_{\mathcal{K}}(\mathrm{x})+\mathrm{Tx}-b$, where*

$$\mathrm{T} = \begin{bmatrix} 5 & 1 \\ 1 & 0 \end{bmatrix} \qquad \textit{and} \qquad \mathrm{b} = \begin{bmatrix} 13 \\ 3 \end{bmatrix}.$$

Note that T *is symmetric and* $\|\mathrm{T}^{-1}\| = 5.1926\ldots$. *When the considered dimension is 2,* $\mathrm{V}(\mathrm{x}) \in \partial_B \mathrm{P}_{\mathcal{K}}(\mathrm{x})$ *is equal to*

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \text{or} \qquad \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Therefore, the matrices $\mathrm{V}(\mathrm{x}) + \mathrm{T}$ *are invertible, for all* $\mathrm{x} \in \mathbb{R}^2$. *Moreover,* $\mathrm{x}^* = [2 \quad 1]^\top$ *is a zero of* F. *By applying Newton method starting at* $\mathrm{x}^0 = [0 \quad 1]^\top$, *for finding the zeros of* F, *the generated sequence oscillates between the points*

$$\mathrm{x}^1 = \begin{bmatrix} 4 \\ -6 \end{bmatrix}, \qquad \mathrm{x}^2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

It is useful to mention that, for all k, $x_1 \neq \pm|\mathrm{x}_2|$. *Therefore,* $\mathrm{V}(\mathrm{x}^k) = \mathrm{P}'_{\mathcal{K}}(\mathrm{x}^k)$ *and this example holds up for all different options in items* **(b)**, **(c)** *and* **(d)** *in* Lemma 4.

3 Application to the linear second order cone complementarity problem

In this section, we apply the results of Section 2 to solve (3) and consequently to find a solution of (4). We begin by showing that from each solution of the semi-smooth equation

$$[\mathrm{M} - \mathrm{Id_n}]\mathrm{P}_{\mathcal{K}}(\mathrm{x}) + \mathrm{x} = -\mathrm{q} \tag{19}$$

we obtain a solution of the LSOCCP (3):

Proposition 4. *If the vector* x^* *is a solution of* (19), *then* $\mathrm{P}_{\mathcal{K}}(\mathrm{x}^*)$ *is a solution of* (3).

Proof. From (5) it follows that $\mathrm{P}_{\mathcal{K}}(\mathrm{x}^*) - \mathrm{x}^* = \mathrm{P}_{\mathcal{K}}(-\mathrm{x}^*)$. Thus, if $\mathrm{x}^* \in \mathbb{R}^n$ is a solution of (19), then

$$\mathrm{M}\mathrm{P}_{\mathcal{K}}(\mathrm{x}^*) + \mathrm{q} = \mathrm{P}_{\mathcal{K}}(-\mathrm{x}^*). \tag{20}$$

Since the second equality in (5) implies that $\langle \mathrm{P}_{\mathcal{K}}(\mathrm{x}^*), \mathrm{P}_{\mathcal{K}}(-\mathrm{x}^*)\rangle = 0$ and since $\mathrm{P}_{\mathcal{K}}(-\mathrm{x}^*) \in \mathcal{K}$, the equality (20) implies that

$$\mathrm{M}\mathrm{P}_{\mathcal{K}}(\mathrm{x}^*) + \mathrm{q} \in \mathcal{K}, \qquad \langle \mathrm{M}\mathrm{P}_{\mathcal{K}}(\mathrm{x}^*) + \mathrm{q},\ \mathrm{P}_{\mathcal{K}}(\mathrm{x}^*)\rangle = 0. \tag{21}$$

Combining this with $\mathrm{P}_{\mathcal{K}}(\mathrm{x}^*) \in \mathcal{K}$, we conclude that $\mathrm{P}_{\mathcal{K}}(\mathrm{x}^*)$ is a solution of (3) as claimed. □

The *semi-smooth Newton method of starting point* $\mathrm{x}^0 \in \mathbb{R}^n$ *for solving* (19), is given by

$$\Big[[\mathrm{M} - \mathrm{Id_n}]\,\mathrm{V}(\mathrm{x}^k) + \mathrm{Id_n}\Big]\,\mathrm{x}^{k+1} = -\mathrm{q}, \qquad \mathrm{V}(\mathrm{x}^k) \in \partial_B \mathrm{P}_{\mathcal{K}}(\mathrm{x}^k), \qquad k = 0, 1, \ldots. \tag{22}$$

Remark 2. *If* $\mathrm{M} - \mathrm{Id_n}$ *is nonsingular, then letting* $\mathrm{T} = [\mathrm{M} - \mathrm{Id_n}]^{-1}$ *and* $b = -\mathrm{T}\mathrm{q}$, *the equation* (1) *becomes* (19). *As a consequence,* (14) *turns into* (22). *Indeed,*

$$\mathrm{x}^{k+1} = \Big[\mathrm{V}(\mathrm{x}^k) + \mathrm{T}\Big]^{-1} b = \Big[[\mathrm{M} - \mathrm{Id_n}]\,\mathrm{V}(\mathrm{x}^k) + \mathrm{Id_n}\Big]^{-1}(-\mathrm{q}), \qquad \mathrm{V}(\mathrm{x}^k) \in \partial_B \mathrm{P}_{\mathcal{K}}(\mathrm{x}^k),$$

for $k = 0, 1, \ldots$, *which is equivalent to the semi-smooth Newton method defined in* (22).

In the following two results we present sufficient existence and uniqueness conditions for (19).

Proposition 5. *If* $\|\mathrm{M}-\mathrm{Id_n}\|<1$ *then* (19) *has a unique solution, for any* $\mathrm{q}\in\mathbb{R}^n$*.*

Proof. First note that, (19) has a unique solution if, and only if, $\Phi(\mathrm{x})=-\left[\mathrm{M}-\mathrm{Id_n}\right]\mathrm{P}_{\mathcal{K}}(\mathrm{x})-\mathrm{q}$ has a unique fixed point. The definition of Φ implies that

$$\Phi(\mathrm{x})-\Phi(\mathrm{y})=\left[\mathrm{M}-\mathrm{Id_n}\right]\left(\mathrm{P}_{\mathcal{K}}(\mathrm{y})-\mathrm{P}_{\mathcal{K}}(\mathrm{x})\right),\qquad \mathrm{x},\mathrm{y}\in\mathbb{R}^n.$$

Hence, from Remark 1 we obtain that $\|\Phi(\mathrm{x})-\Phi(\mathrm{y})\|\leq\|\mathrm{M}-\mathrm{Id_n}\|\,\|\mathrm{x}-\mathrm{y}\|$. Since $\|\mathrm{M}-\mathrm{Id_n}\|<1$, Φ is a contraction. Therefore, from Theorem 1 we conclude that Φ has a unique fixed point, for any $\mathrm{q}\in\mathbb{R}^n$, which implies that (19) has a unique solution. □

Proposition 6. *If* M *is nonsingular and* $\left\|\mathrm{M}^{-1}-\mathrm{Id_n}\right\|<1$*, then* (19) *has a unique solution, for any* $\mathrm{q}\in\mathbb{R}^n$*.*

Proof. Since M is nonsingular, by taking into account the first equality in (5), after some algebraic manipulations we can conclude that (19) is equivalent to

$$\left[\mathrm{M}^{-1}-\mathrm{Id_n}\right]\mathrm{P}_{\mathcal{K}}(-\mathrm{x})-\mathrm{x}=\mathrm{M}^{-1}\mathrm{q}. \tag{23}$$

Define the auxiliary function $\Theta(\mathrm{x})=\left[\mathrm{M}^{-1}-\mathrm{Id_n}\right]\mathrm{P}_{\mathcal{K}}(-\mathrm{x})-\mathrm{M}^{-1}\mathrm{q}$. Note that (23) has a unique solution if, and only if, Θ has a unique fixed point. On the other hand, the definition of Θ implies

$$\Theta(\mathrm{x})-\Theta(\mathrm{y})=\left[\mathrm{M}^{-1}-\mathrm{Id_n}\right]\left(\mathrm{P}_{\mathcal{K}}(-\mathrm{x})-\mathrm{P}_{\mathcal{K}}(-\mathrm{y})\right),\qquad \mathrm{x},\mathrm{y}\in\mathbb{R}^n.$$

It follows from Remark 1 that $\|\Theta(\mathrm{x})-\Theta(\mathrm{y})\|\leq\left\|\mathrm{M}^{-1}-\mathrm{Id_n}\right\|\|\mathrm{x}-\mathrm{y}\|$ and, due to $\left\|\mathrm{M}^{-1}-\mathrm{Id_n}\right\|<1$, we conclude that Θ is a contraction. Therefore, Theorem 1 implies that Θ has a unique fixed point, for any $\mathrm{q}\in\mathbb{R}^n$. Consequently, (19) has a unique solution, for any $\mathrm{q}\in\mathbb{R}^n$. □

Remark 3. *Note that, there exist symmetric matrices for which neither* $\|\mathrm{M}-\mathrm{Id_n}\|<1$*, nor* $\left\|\mathrm{M}^{-1}-\mathrm{Id_n}\right\|<1$ *are satisfied. For example, such a matrix is*

$$\mathrm{M}=\begin{bmatrix}1/3 & 0\\ 0 & 3\end{bmatrix}. \tag{24}$$

In the next theorem a convergence result for the semi-smooth Newton sequence $\{\mathrm{x}^k\}$, generated by (22), is presented.

Theorem 4. *Let* $q\in\mathbb{R}^n$ *and* $\mathrm{M}\in\mathbb{R}^{n\times n}$ *be a symmetric matrix. Assume that* $\mathrm{M}-\mathrm{Id_n}$ *is nonsingular and* $\|\mathrm{M}-\mathrm{Id_n}\|<1$*. Then,* (19) *has a unique solution* $\mathrm{x}^*\in\mathbb{R}^n$ *and, for any starting point* $\mathrm{x}^0\in\mathbb{R}^n$*, the sequence* $\{\mathrm{x}^k\}$ *generated by* (22) *is well-defined. Additionally, if* $\|\mathrm{M}-\mathrm{Id_n}\|<1/2$*, then* $\{\mathrm{x}^k\}$ *converges Q-linearly to* $\mathrm{x}^*\in\mathbb{R}^n$*, the unique solution of* (1)*, as follows:*

$$\|\mathrm{x}^*-\mathrm{x}^{k+1}\|\leq\frac{\|\mathrm{M}-\mathrm{Id_n}\|}{1-\|\mathrm{M}-\mathrm{Id_n}\|}\|\mathrm{x}^*-\mathrm{x}^k\|,\qquad k=0,1,\ldots.$$

Moreover, $\mathrm{P}_{\mathcal{K}}(\mathrm{x}^*)$ *is a solution of* (3)*.*

Proof. The proof follows by combining Proposition 5, Remark 2, Theorem 2, and Proposition 4. □

If we assume M is a symmetric and positive definite in Theorem 4 then stronger results are obtained. We begin showing that the semi-smooth Newton method in (22) is always well-defined.

Lemma 7. *If* M *is symmetric and positive definite, then the following matrix is nonsingular*

$$[\mathrm{M} - \mathrm{Id_n}]\,\mathrm{V}(\mathrm{x}) + \mathrm{Id_n}, \qquad \mathrm{x} \in \mathbb{R}^n. \tag{25}$$

As a consequence, the semi-smooth Newton sequence $\{\mathrm{x}^k\}$ *generated by* (22) *is well-defined, for any starting point* $\mathrm{x}^0 \in \mathbb{R}^n$*.*

Proof. To simplify the notations let $\mathrm{V} = \mathrm{V}(\mathrm{x})$. Let us suppose, by contradiction, that the matrix in (25) is singular. Thus there exists $\mathrm{u} \in \mathbb{R}^n$ such that

$$([\mathrm{M} - \mathrm{Id_n}]\,\mathrm{V} + \mathrm{Id_n})\,\mathrm{u} = 0, \qquad \mathrm{u} \neq 0.$$

It is straightforward to see that the last equality is equivalent to the following one

$$\mathrm{MVu} = [\mathrm{V} - \mathrm{Id_n}]\,\mathrm{u}, \qquad \mathrm{u} \neq 0. \tag{26}$$

Since M is symmetric and positive definite, there exists a nonsingular matrix $\mathrm{L} \in \mathbb{R}^{n\times n}$ such that $\mathrm{M} = \mathrm{LL}^\top$. Taking into account that $\mathrm{M} = \mathrm{LL}^\top$ and that Lemma 4 implies $\mathrm{V} = \mathrm{V}^\top$ and $\|\mathrm{V}\| \leq 1$, the equality in (26) easily implies that

$$\left\|\mathrm{L}^\top \mathrm{Vu}\right\|^2 = \langle \mathrm{VMVu}, \mathrm{u}\rangle = \left\langle (\mathrm{V}^2 - \mathrm{V})\mathrm{u}, \mathrm{u}\right\rangle \leq 0.$$

Thus, $\mathrm{L}^\top\mathrm{Vu} = 0$. Because $\mathrm{M} = \mathrm{LL}^\top$ and $\mathrm{L}^\top\mathrm{Vu} = 0$, equality (26) implies that $(\mathrm{V} - \mathrm{Id_n})\mathrm{u} = 0$, or equivalently, $\mathrm{Vu} = \mathrm{u}$. Hence,

$$\mathrm{L}^\top\mathrm{u} = \mathrm{L}^\top\mathrm{Vu} = 0, \qquad \mathrm{u} \neq 0,$$

which contradicts the nonsingularity of L. Therefore, the matrix in (25) is nonsingular for all $\mathrm{x} \in \mathbb{R}^n$ and the first part of the lemma is proven.

To prove the second part of the lemma, combine the formal definition of $\{\mathrm{x}^k\}$ in (22) and the first part of this lemma. □

Theorem 5. *Let* $\mathrm{q} \in \mathbb{R}^n$ *and* $\mathrm{M} \in \mathbb{R}^{n\times n}$ *be a symmetric and positive definite matrix. Then, for any starting point* $\mathrm{x}^0 \in \mathbb{R}^n$*, the semi-smooth Newton sequence* $\{\mathrm{x}^k\}$ *generated by* (22) *is well-defined. Additionally, if* $\mathrm{M} - \mathrm{Id_n}$ *is positive definite and* $\|\mathrm{M} - \mathrm{Id_n}\| < 1$*, then* (19) *has a unique solution* $\mathrm{x}^* \in \mathbb{R}^n$ *and* $\{\mathrm{x}^k\}$ *converges Q-linearly to* x^* *as follows*

$$\|\mathrm{x}^* - \mathrm{x}^{k+1}\| \leq \|\mathrm{M} - \mathrm{Id_n}\|\,\|\mathrm{x}^* - \mathrm{x}^k\|, \qquad k = 0, 1, \ldots.$$

Moreover, $\mathrm{P}_{\mathcal{K}}(\mathrm{x}^*)$ *is a solution of* (3)*.*

Proof. The proof follows by combining Proposition 5, Lemma 7, Remark 2, Theorem 3 and Proposition 4. □

From now on, we will consider a parametric version of equation (19), which will be specially useful to study the second order cone linear complementarity problem, whenever M is positive definite. Let $q \in \mathbb{R}^n$ and $\mathrm{M} \in \mathbb{R}^{n\times n}$ be a symmetric matrix defining equation in (19). Let $\beta > 0$ and define $\mathrm{M}_\beta := \beta\mathrm{M}$, $\mathrm{q}_\beta := \beta\mathrm{q}$ and consider the *parametric auxiliary equation*

$$[\mathrm{M}_\beta - \mathrm{Id_n}]\mathrm{P}_{\mathcal{K}}(\mathrm{y}) + \mathrm{y} = -\mathrm{q}_\beta. \tag{27}$$

Note that the last equation has the same algebraic structure of equation in (19). In the next remark we point out some properties of (27), which are analogous proporties of equation in (19).

Remark 4. *It is worth mentioning that the result of* Proposition 4 *remains true if equation* (19) *is replaced by* (27)*. In other words, if* y^* *is solution of* (27)*, then* $\mathrm{P}_{\mathcal{K}}(\mathrm{y}^*)$ *is a solution of* (3)*. The proof of this statement follows from the same idea as in the proof of* Proposition 4*, by noting that, due to* $\mathcal{K}$ *being a cone, the equation* (21) *still holds for* $\mathrm{M} = \mathrm{M}_\beta$ *and* $\mathrm{q} = \mathrm{q}_\beta$*. Moreover, if* $\|\mathrm{M}_\beta - \mathrm{Id_n}\| < 1$ *or* $\left\|{\mathrm{M}_\beta}^{-1} - \mathrm{Id_n}\right\| < 1$*, then equation in* (27) *has also a unique solution. Indeed, the result follows by applying* Propositions 5 *and* 6 *with* $\mathrm{M} = \mathrm{M}_\beta$ *and* $\mathrm{q} = \mathrm{q}_\beta$*.*

Now we are going to show the advantage to choose an appropriate parameter $\beta > 0$ in (27) instead of taking $\beta = 1$ as in equation (19). We will begin with the following remark.

Remark 5. *Additionally, if* M *is positive definite and* $0 \leq \beta < 2/\|\mathrm{M}\|$*, which includes the simple example of* Remark 3*, then equation* (27) *always has a unique solution. Actually, if* M *is positive definite and* $0 < \beta < 2/\|\mathrm{M}\|$*, then we have*

$$\|\mathrm{M}_\beta - \mathrm{Id_n}\| = \|\beta\mathrm{M} - \mathrm{Id_n}\| < 1,$$

and, by applying Proposition 5 *with* $M = \mathrm{M}_\beta$ *and* $\mathrm{q} = \mathrm{q}_\beta$*, we conclude that* (27) *has a unique solution. In particular, note that, by replacing the matrix in* (24)*, which has the norm equal to* 3*, with the matrix*

$$\mathrm{M}_\beta := \begin{bmatrix} \beta/3 & 0 \\ 0 & 3\beta \end{bmatrix}, \qquad 0 < \beta < 1/3,$$

we have $\|\mathrm{M}_\beta - \mathrm{Id_n}\| < 1$ *and in this case we conclude that* (27) *has a unique solution.*

The semi-smooth Newton method for solving (27), with starting point $\mathrm{y}^0 \in \mathbb{R}^n$, is given by

$$\Big[[\mathrm{M}_\beta - \mathrm{Id_n}]\,\mathrm{V}(\mathrm{y}^k) + \mathrm{Id_n}\Big]\,\mathrm{y}^{k+1} = -\beta\mathrm{q}, \qquad \mathrm{V}(\mathrm{y}^k) \in \partial_B \mathrm{P}_{\mathcal{K}}(\mathrm{y}^k), \qquad k = 0, 1, \ldots. \tag{28}$$

The next result shows how to take advantage of choosing an appropriate parameter $\beta > 0$, in order to apply the semi-smooth Newton method for obtaining a solution of (3).

Theorem 6. *Let* $q \in \mathbb{R}^n$ *and* $\mathrm{M} \in \mathbb{R}^{n\times n}$ *be a symmetric positive definite matrix. Then, for any starting point* $\mathrm{y}^0 \in \mathbb{R}^n$*, the semi-smooth Newton sequence* $\{\mathrm{y}^k\}$ *generated by* (28) *is well-defined. Moreover, if* $0 < \beta < 2/\|\mathrm{M}\|$ *then* (27) *has a unique solution* $\mathrm{y}^* \in \mathbb{R}^n$*. Additionally, if* $\|\mathrm{M}\|\|\mathrm{M}^{-1}\| < 3$ *and*

$$\frac{1}{2}\|\mathrm{M}^{-1}\| < \beta < \frac{3}{2}\frac{1}{\|\mathrm{M}\|}, \tag{29}$$

then the sequence $\{\mathrm{y}^k\}$ *converges Q-linearly to* $\mathrm{y}^* \in \mathbb{R}^n$*, the unique solution of* (27)*, as follows:*

$$\|\mathrm{y}^* - \mathrm{y}^{k+1}\| \leq \frac{\|\mathrm{M}_\beta - \mathrm{Id_n}\|}{1 - \|\mathrm{M}_\beta - \mathrm{Id_n}\|}\|\mathrm{y}^* - \mathrm{y}^k\|, \qquad k = 0, 1, \ldots. \tag{30}$$

Furthermore, $\mathrm{P}_{\mathcal{K}}(\mathrm{y}^*)$ *is a solution of* (3)*.*

Proof. By using the same idea as in Lemma 7, we can prove that $[\mathrm{M}_\beta - \mathrm{Id_n}]\,\mathrm{V}(\mathrm{y}^k) + \mathrm{Id_n}$ is a nonsingular matrix, for $k = 0, 1, \ldots$. Consequently, for any starting point $\mathrm{y}^0 \in \mathbb{R}^n$, the semi-smooth Newton sequence $\{\mathrm{y}^k\}$ generated by (28) is well-defined. Now, assuming that $0 < \beta < 2/\|\mathrm{M}\|$, we conclude from Remark 5 that equation in (27) has a unique solution $\mathrm{y}^* \in \mathbb{R}^n$. Then,

$$[[\mathrm{M}_\beta - \mathrm{Id_n}]\,\mathrm{V}(\mathrm{y}^*) + \mathrm{Id_n}]\,\mathrm{y}^* = -\beta\mathrm{q},$$

which, together with definition of $\{y^k\}$ in (28) and $V(x)x = P_{\mathcal{K}}(x)$ for all $x \in \mathbb{R}^n$, yield

$$\begin{aligned} y^{k+1} - y^* &= \Big[[M_\beta - \mathrm{Id_n}]\, V(y^k) + \mathrm{Id_n}\Big]^{-1} \Big[[[M_\beta - \mathrm{Id_n}]\, V(y^*) + \mathrm{Id_n}]\, y^* - \Big[[M_\beta - \mathrm{Id_n}]\, V(y^k) + \mathrm{Id_n}\Big]\, y^*\Big] \\ &= \Big[[M_\beta - \mathrm{Id_n}]\, V(y^k) + \mathrm{Id_n}\Big]^{-1} [M_\beta - \mathrm{Id_n}] \Big[P_{\mathcal{K}}(y^*) - P_{\mathcal{K}}(y^k) - V(y^k)\Big(y^* - y^k\Big)\Big], \end{aligned}$$

for $k = 0, 1, \ldots$. By combining Lemma 6 with this equality and by using the properties of the norm, we have

$$\|y^* - y^{k+1}\| \le \left\| \Big[[M_\beta - \mathrm{Id_n}]\, V(y^k) + \mathrm{Id_n}\Big]^{-1} \right\| \|M_\beta - \mathrm{Id_n}\|\, \|y^* - y^k\|. \tag{31}$$

On the other hand, for $0 < \beta < 2/\|M\|$, we have $\big\|[M_\beta - \mathrm{Id_n}]\, V(y^k)\big\| \le \|M_\beta - \mathrm{Id_n}\| < 1$. Thus, by using Lemma 1, we have

$$\left\| \Big[[M_\beta - \mathrm{Id_n}]\, V(y^k) + \mathrm{Id_n}\Big]^{-1} \right\| \le \frac{1}{1 - \|M_\beta - \mathrm{Id_n}\|},$$

which combined with (31) gives (30). Moreover, by using assumption (29), we conclude

$$\frac{\|M_\beta - \mathrm{Id_n}\|}{1 - \|M_\beta - \mathrm{Id_n}\|} < 1.$$

Thus, the last inequality, together with (30), imply that $\{y^k\}$ converges Q-linearly to y^* and, by using Remark (4), we obtain that $P_{\mathcal{K}}(y^*)$ is a solution of (3). □

We end this section by presenting an upper bound for the rate of convergence of the semi-smooth Newton method in (28), which depends on M only.

Remark 6. *Let us focus our attention on the convergence rate of $\{y^k\}$, the sequence generated by the semi-smooth Newton method in* (28)*, when* M *is symmetric and positive definite. The inequality in* (30) *shows that $\|M_\beta - \mathrm{Id_n}\|$ determines the rate, which depends on β. Indeed, the upper bound for the rate of convergence is*

$$r(\beta) := \frac{\|\beta M - \mathrm{Id_n}\|}{1 - \|\beta M - \mathrm{Id_n}\|}, \qquad \frac{1}{2}\|M^{-1}\| < \beta < \frac{3}{2}\frac{1}{\|M\|}.$$

Now, we are going to compute the minimum value of the function r in the range of β given above. Since the function $t \mapsto t/(1-t)$ is increasing, the minimum value of r in this range is reached when

$$\beta_* = \operatorname{argmin}\left\{ \|\beta M - \mathrm{Id_n}\| \ : \ \frac{1}{2}\|M^{-1}\| < \beta < \frac{3}{2}\frac{1}{\|M\|} \right\}. \tag{32}$$

Let $\lambda_{\min}$ and $\lambda_{\max}$ be the minimum and the maximum eigenvalues of M*, respectively. Thus, since* M *is symmetric and positive definite $\|M^{-1}\| = 1/\lambda_{\min}$ and $\lambda_{\max} = \|M\|$, and moreover, by using* (32)*, we obtain*

$$\beta_* = \operatorname{argmin}\left\{ \max\{|\beta\lambda_{\min} - 1|, |\beta\lambda_{\max} - 1|\} \ : \ \frac{1}{2}\frac{1}{\lambda_{\min}} < \beta < \frac{3}{2}\frac{1}{\lambda_{\max}} \right\}.$$

Some calculations show that

$$\beta_* = \frac{2}{\lambda_{\max} + \lambda_{\min}}, \qquad r(\beta_*) = \frac{\lambda_{\max} - \lambda_{\min}}{2\lambda_{\min}}.$$

Additionally, if $\lambda = \lambda_{\max} = \lambda_{\min}$, then $\beta_* = 1/\lambda$ and $r(\beta_*) = 0$. Thus, the inequality in (30) *implies that $y^1 = y^*$. Hence the sequence $\{y^k\}$ generated by* (28) *with $\beta = \beta_*$ converges to y^* in just one iteration.*

4 Computational Results

We implemented the semi-smooth Newton method (14) for solving equation (1) in Matlab 7.11.0.584 (R2010b). When the projection mapping $\mathrm{P}_{\mathcal{K}}$ onto the second order cone $\mathcal{K}$ is not continuously differentiable at $\mathrm{x} \in \mathbb{R}^n$, we define $\mathrm{V(x)} \in \partial_B \mathrm{P}_{\mathcal{K}}(\mathrm{x})$ in the simplest way. This means that $\mathrm{V(x)}$ is equal to $\mathrm{Id_n}$ in case of item (a) and the null matrix in cases of items (b) and (c) of Lemma 4. The method stops at the iterate $\mathrm{x}^k \in \mathbb{R}^n$ reporting "Solution found" if $\left\|\mathrm{P}_{\mathcal{K}}(\mathrm{x}^k) + \mathrm{T}\mathrm{x}^k - b\right\| \leq 10^{-6}$. Failure is considered when the number of iterations exceeds 20. All codes are freely available at `https://orizon.mat.ufg.br/p/3374-links`. The experiments were run on a 3.4 GHz Intel(R) i7 with 4 processors, 8Gb of RAM, and Linux operating system.

In order to verify the applicability of our approach, we tested the semi-smooth Newton method (14) in several random problems (1). A linear system must be solved in each iteration of the method. For this purpose, we used the *mldivide* (same as *backslash*) command of Matlab. Following, we enlighten how the problems data were generated.

(i) *Matrix* T: We consider cases where the matrix T is dense and cases where T is sparse for different dimension values n. In the first case, we randomly generated the fully dense matrix T from a uniform distribution on $(-10, 10)$. To ensure the fulfillment of the hypothesis (15), we computed the minimum singular value of T, then we rescaled T by multiplying it by 2 divided by the minimum singular value multiplied by a random number in the interval $(0, 1)$. To construct a sparse matrix T we used the Matlab routine *sprand*, which generates a sparse matrix with predefined dimension, density and singular values. First, we randomly generated the vector of singular values from a uniform distribution on $(0, 1)$. The fulfillment of the hypothesis (15) can be easily achieved by conveniently rescaling the singular values. Finally, we evoke *sprand* with density equal to 0.004. This means that, only about 0.4% of the elements of T are non null.

(ii) *Solution and vector b*: By Proposition 2, equation (1) has a unique solution if $\|\mathrm{T}^{-1}\| < 1$. Note that if $\mathrm{x}^* = (x_1^*, \mathrm{x}_2^*) \in \mathbb{R} \times \mathbb{R}^{n-1}$ with $x_1^* \leq -\|\mathrm{x}_2^*\|$ being a solution of (1), then x^* is a solution of $\mathrm{Tx} = b$. On the other hand, if $x_1^* \geq \|\mathrm{x}_2^*\|$, then x^* is a solution of $[\mathrm{Id_n} + \mathrm{T}]\mathrm{x} = \mathrm{b}$. In both cases, the solution can be found by simply solving a linear system. We ignore these trivial cases by assuming that the unique solution x^* of (1) is such that $-\|\mathrm{x}_2^*\| < x_1^* < \|\mathrm{x}_2^*\|$. First, we randomly generated $\mathrm{x}_2^* \in \mathbb{R}^{n-1}$ from a uniform distribution on $(-10, 10)$ and then we defined $x_1^* \in \mathbb{R}$ as a convex combination between $-\|\mathrm{x}_2^*\|$ and $\|\mathrm{x}_2^*\|$. After that, we computed $b = \mathrm{P}_{\mathcal{K}}(\mathrm{x}^*) + \mathrm{T}\mathrm{x}^*$.

(iii) *Initial point*: As preliminary numerical tests, we investigated the influence of the starting point in the performance of the method. At this stage, we generated 100 problems with fully dense 1000×1000 matrix T and 100 problems with sparse 5000×5000 matrix T. For each problem, we ran the semi-smooth Newton method starting from different initial points $\mathrm{x}^0 = (x_1^0, \mathrm{x}_2^0) \in \mathbb{R} \times \mathbb{R}^{n-1}$ such that $x_1^0 > \|\mathrm{x}_2^0\|$, $x_1^0 < -\|\mathrm{x}_2^0\|$ and $-\|\mathrm{x}_2^0\| < x_1^0 < \|\mathrm{x}_2^0\|$. Observe that these regions are the interior of the cone $\mathcal{K}$, the interior of the polar cone $\mathcal{K}^\circ$ and the interior of the complement of $\mathcal{K} \cup \mathcal{K}^\circ$, respectively. For simplicity, let us call these regions by *Region 1, 2* and *3*, respectively. For dense instances, the method presented similar performance (in the sense of number of problems solved and average CPU time required) regardless of the location of the initial point. For sparse instances, the average CPU time to solve the problems was 8.61s, 8.14s, and 9.22s for the initial point in Region 1, 2, and 3, respectively. Let us explain this slight difference. When T is dense, the matrix of the linear system $[\mathrm{T} + \mathrm{V}(\mathrm{x}^0)]\mathrm{x} = \mathrm{b}$ solved at the first iteration is also dense regardless of the ocation of the initial point. On the other hand, for sparse T, the matrix of the linear system $[\mathrm{T} + \mathrm{V}(\mathrm{x}^0)]\mathrm{x} = \mathrm{b}$ is sparse if x^0 belongs to Region 1 or 2, and dense if x^0 belongs to Region 3, see Lemma 4. Since the method requires very few iterations to find the solution, the computational cost of the first iterations justify the difference between the average CPU times. At this stage, we conclude that it is advantageous to take the starting point into Region 1 or 2. If x^0 belongs to Region 2, then $V(x^0) = 0$ and the first iterate x^1 is the unique solution of the linear system $Tx = b$. For simplicity, we assume directly that the starting point is given by the unique solution of $Tx = b$.

For dense instances, we consider dimensions $n = 500$, 1000, 2000, and 3000, and for sparse instances $n = 3000$, and 5000. We generate 200 different problems for each test set. In general, when T is a dense matrix, its condition number is of order 10^3 or 10^4. For comparative purposes, the singular values of a sparse matrix T were rescaled so that its condition number is of order 10^4. Table 1 gives a summary of our numerical experiments. The column "n" is the dimension of the test set, "Cond(T)" is the average condition number of matrices T, "Problems solved" informs the number of successfully solved problems, "It" and "time" are the average number of semi-smooth Newton iterations, and the average CPU time for the solved problems, respectively.

	n	Cond(T)	Problems solved	It	time (s)
Dense T	500	1.27×10^4	198 (99.0%)	1.97	0.03
	1000	1.40×10^4	187 (93.5%)	1.97	0.14
	2000	4.83×10^4	140 (70.0%)	2.25	0.79
	3000	4.13×10^4	106 (53.0%)	2.23	2.00
Sparse T	3000	1.92×10^4	194 (97.0%)	1.96	1.75
	5000	1.88×10^4	194 (97.0%)	1.94	6.07

Table 1: Performance of semi-smooth Newton method in sets of 200 random problems considering fully dense matrices T, and sparse matrices T (density approximately 0.4%).

The semi-smooth Newton method solves a typical problem with two iterations. In fact, considering the 1019 solved problems for all instances, 970 (95.2%) problems were solved with 2 iterations, while 36 (3.5%) problems were solved with 1 iteration and 13 (1.3%) problems were solved with more than 2 iterations. It is interesting to point out that, for any considered problem, *all* iterates x^k belongs to Region 3 (since the solution also belong to this set, this fact is not a big surprise). Therefore, the matrix $T + V(x^k)$ is dense and the command *mldivide* of Matlab uses a LU solver for the associated linear system $[T + V(x^k)]x = b$.

The robustness of the semi-smooth Newton method is directly connected to the ability of the linear system solver used. It is rarely true in practical implementations that direct methods for linear systems give the exact solution. In some cases, they are not able to find the solution with high accuracy and the convergence of the main method gets impaired. By means of numerical observations, we realize that in instances where the command *mldivide* is able to give the solution of a linear system with residuum less than 10^{-6}, the semi-smooth Newton method stops reporting "Solution found". Otherwise, the semi-smooth Newton method is not able to find the solution with the desired accuracy. Studies concerning the convergence theory of an inexact Newton method are necessary to clarify these issues. As we can see in Table 1, in case of dense matrices T, the number of problems solved by the semi-smooth Newton method decreases according to the increase of the dimension n. This phenomenon is clearly connected with the fact that the greater the dimension, the more operations are required to solve a linear equation. Consequently, the method is most affected by the accumulation of floating-point errors resulting in lower robustness. However, it is useful to mention that, in all cases of failure the achieved accuracy was close to the desired one (typically, order of 10^{-6}). For sparse T instances, the command *mldivide* is able to solve linear systems with high accuracy and the robustness of the semi-smooth Newton method is not affected.

We close the computational results by testing the method in problems where T is a symmetric and positive definite matrix. In this case, by Theorem 3, equation (1) has a unique solution for any $\mathrm{b} \in \mathbb{R}^n$ and the semi-smooth Newton method (14) is well-defined. Moreover, convergence is guaranteed if $\|\mathrm{T}^{-1}\| < 1$. We randomly generated 200 problems with 1000×1000 matrices T that do *not* fulfill this hypothesis. Let us clarify this. First, we randomly generated a fully dense matrix A and a vector $\alpha \in \mathbb{R}^n$ of eigenvalues of T from a uniform distribution on $(0, 1)$. After that, we extracted the eigenvectors of matrix $(\mathrm{A} + \mathrm{A}^\top)/2$ in the columns of a matrix U and defined $\mathrm{T} = \mathrm{UDU}^\top$, where D is the diagonal matrix with (i, i)-th entry equal to α_i, $i = 1, \ldots, 1000$. Finally, we randomly generated the vector b as in the previous experiments. The average value of $\|\mathrm{T}^{-1}\|$ was 4.90×10^3. The semi-smooth Newton method successfully solved *all* problems of this test set. The average number of iterations and the average CPU time for solving the problems were 5.90 and 0.39 seconds, respectively. We observed that, as in the previous tests, if we rescaled the eigenvalues of T such that hypothesis (15) was fulfilled, the semi-smooth Newton method required (in general) two iterations for finding the solution with the desired accuracy. When T is a symmetric and positive definite matrix, this experiment suggests the conjecture that the semi-smooth Newton method always converges.

5 Final remarks

In this paper we studied a special equation associated to the second order cone. Our main result shows that, under mild conditions, we can apply a semi-smooth Newton method for finding a solution of this equation. Besides, the generated sequence converges globally and linearly to the solution. Our numerical tests suggest that the semi-smooth Newton method always converges if the matrix of the equation is symmetric and positive definite. The equation is important because it is related to linear second order cone complementarity problems, used in a wide range of applications [21]. It would be interesting to see whether the used technique can be applied for solving a similar equation associated to the cone of the semidefinite matrices. A more general open question is whether our semi-smooth Newton method approach can be unified to solve linear symmetric cone complementarity problems. The importance of this question is due to its connections with physics, mechanics, economics, game theory, robotics, optimization and neural networks, such as the ones described in [1, 2, 10, 19, 22, 27, 28, 33, 34]. We remark that any quadratic second order symmetric cone optimization problem can be reformulated in terms of linear complementarity problems related to symmetric cones, a further motivation to study this question. We foresee further progress in this topic in the near future.

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