

Proximal point method for a special class of nonconvex functions on Hadamard manifolds

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In this article, we present the proximal point method for finding minima of a special class of nonconvex function on a Hadamard manifold. The well definedness of the sequence generated by the proximal point method is established. Moreover, it is proved that each accumulation point of this sequence satisfies the necessary optimality conditions and, under additional assumptions, its convergence for a minima is obtained.

Keywords: proximal point method; nonconvex functions; Hadamard manifolds

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1. Introduction

The extension of the concepts and techniques of the Mathematical Programming from Euclidean space to Riemannian manifolds occurs naturally and has been frequently done in recent years, with a theoretical purpose as well as to obtain effective algorithms, see [1–10,12–14,16–20,25–31,38–41,43,48,49,52–56]. In particular, we observe that these extensions allow the solving of some nonconvex constrained problems in Euclidean space. More precisely, nonconvex problems in the classic sense may become convex with the introduction of an adequate Riemannian metric on the manifold (see, e.g. [10,14]). The proximal point algorithm, introduced by Martinet [32] and Rockafellar [44], has been extended to different contexts, see [5,18,27,40] and the references therein. In [18], the proximal point method has been generalized in order to solve convex optimization problems of the form

$$\begin{aligned} (P) \quad & \min f(p) \\ & \text{s.t. } p \in M, \end{aligned} \tag{1}$$

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where M is a Hadamard manifold and $f: M \rightarrow \mathbb{R}$ is a convex function (in Riemannian sense). The method was described as follows:

$$p^{k+1} := \operatorname{argmin}_{p \in M} \left\{ f(p) + \frac{\lambda_k}{2} d^2(p, p^k) \right\}, \quad (2)$$

with $p^\circ \in M$ an arbitrary point, d the intrinsic Riemannian distance (to be defined later on) and $\{\lambda_k\}$ a sequence of positive numbers. The authors also showed that this extension is natural. In [27], the important notion of maximal monotonicity from a multivalued operator defined on a Banach space to multivalued vector field defined on a Hadamard manifold has been extended. Besides, the authors present a general proximal point method to finding singularity of a multivalued vector field. In particular, as an application of the convergence result obtained for the proposed algorithm, constrained optimization problems have been solved. With regard to [40] the authors generalized the proximal point method with Bregman distance for solving quasiconvex and convex optimization problems on Hadamard manifolds. Spingarn has in [48], in particular, developed the proximal point method for the minimization of a certain class of nondifferentiable and nonconvex functions, namely, lower- C^2 functions defined on the Euclidean space, see also [21]. Kaplan and Tichatschke in [24] also applied the proximal point method for the minimization of a similar class of the ones studied in [21,48], namely, the maximum of continuously differentiable functions.

Our goal is to study the same class objective functions studied by Kaplan and Tichatschke [24], in the Riemannian context, applying the proximal point method (2) in order to solve the problem (1) with the objective function in that class. For this purpose, it is necessary to study the generalized directional derivative and subdifferential in the Riemannian manifolds context. Several works have studied such concepts and presented many useful results in the Riemannian nonsmooth optimization context, see for example [4,26,33,50].

The organization of this article is as follows. In Section 1.1 we define the notations and list some results of Riemannian geometry to be used throughout this article. In Section 2, we recall some facts of the convex analysis on Hadamard manifolds. In Section 3, we present some properties of the directional derivative of a convex function defined on a Hadamard manifold, including a characterization of the directional derivative and of the subdifferential of the maximum of a certain class of convex functions. Here, we also present the definition for the generalized directional derivative of locally Lipschitz functions (not necessarily convex) and an important property of the subdifferential of the maximum of differentiable continuously functions. In Section 4, we present an application of the proximal point method (2) to solve the problem (1) in the case where the objective function is a real-valued function on a Hadamard manifold M (not necessarily convex) given by the maximum of a certain class of functions. The main results are the proof of the well definedness of the sequence generated by (2), the proof that each accumulation point of this sequence is a stationary point of the objective function and, under some additional assumptions, the proof of convergence of that sequence to a solution of the problem (1). Finally in Section 5 we provide two examples, in one of them the curvature of the manifold in consideration is identically zero and the other one is nonpositive but not identically zero. In both the examples,

the proximal point method for nonconvex problems, proposed in this article, is applied.

1.1. Notation and terminology

In this section, we introduce some fundamental properties and notations about Riemannian geometry. These basic facts can be found in any introductory book on Riemannian geometry, such as [15,46].

Let M be an n -dimensional connected manifold. We denote by T_pM the n -dimensional *tangent space* of M at p , by $TM = \cup_{p \in M} T_pM$ *tangent bundle* of M and by $\mathcal{X}(M)$ the space of smooth vector fields over M . A family $\langle \cdot, \cdot \rangle$ of inner products $\langle \cdot, \cdot \rangle_p$ on each tangent space T_pM that varies smoothly from point to point (in the sense that if $X, Y \in \mathcal{X}(M)$, then $p \mapsto \langle X(p), Y(p) \rangle_p$ is a smooth function) is called a Riemannian metric. When M is endowed with a Riemannian metric $\langle \cdot, \cdot \rangle$, with the corresponding norm denoted by $\| \cdot \|$, then M is called a Riemannian manifold. Recall that the metric can be used to define the length of piecewise smooth curves $\gamma: [a, b] \rightarrow M$ joining p to q , i.e. such that $\gamma(a) = p$ and $\gamma(b) = q$, by

$$l(\gamma) = \int_a^b \|\gamma'(t)\| dt,$$

and, moreover, by minimizing this length functional over the set of all such curves, we obtain a Riemannian distance $d(p, q)$ inducing the original topology on M . The metric induces a map $f \mapsto \text{grad} f \in \mathcal{X}(M)$ which, for each function smooth over M , associates its gradient via the rule $\langle \text{grad} f, X \rangle = df(X)$, $X \in \mathcal{X}(M)$. Let ∇ be the Levi-Civita connection associated with $(M, \langle \cdot, \cdot \rangle)$. In each point $p \in M$, we have a linear map $A_X(p): T_pM \rightarrow T_pM$ defined by

$$A_X(p)v = \nabla_v X. \tag{3}$$

If $X = \text{grad} f$, where $f: M \rightarrow \mathbb{R}$ is a twice differentiable function, then $A_X(p)$ is the *Hessian* of f at p and is denoted by $\text{Hess}_p f$. A vector field V along γ is said to be *parallel* if $\nabla_{\gamma'} V = 0$. If γ' itself is parallel we say that γ is a *geodesic*. Given that the geodesic equation $\nabla_{\gamma'} \gamma' = 0$ is a second-order nonlinear ordinary differential equation, we conclude that the geodesic $\gamma = \gamma_v(\cdot, p)$ is determined by its position p and velocity v at p . It is easy to check that $\|\gamma'\|$ is constant. We say that γ is *normalized* if $\|\gamma'\| = 1$. The restriction of a geodesic to a closed bounded interval is called a *geodesic segment*. A geodesic segment joining p to q in M is said to be *minimal* if its length is equal to $d(p, q)$ and the geodesic in question is said to be a *minimizing geodesic*. If γ is a geodesic joining points p and q in M then, for each $t \in [a, b]$, ∇ induces a linear isometry, relative to $\langle \cdot, \cdot \rangle$, $P_{\gamma(a)\gamma(t)}: T_{\gamma(a)}M \rightarrow T_{\gamma(t)}M$, the so-called *parallel transport* along γ from $\gamma(a)$ to $\gamma(t)$. The inverse map of $P_{\gamma(a)\gamma(t)}$ is denoted by $P_{\gamma(a)\gamma(t)}^{-1}: T_{\gamma(t)}M \rightarrow T_{\gamma(a)}M$. In the particular case of γ to be the unique geodesic segment joining p and q , then the parallel transport along γ from p to q is denoted by $P_{pq}: T_pM \rightarrow T_qM$.

A Riemannian manifold is *complete* if the geodesics are defined for any values of t . Hopf–Rinow’s theorem asserts that if this is the case then any pair of points, say p and q , in M can be joined by a (not necessarily unique) minimal geodesic segment. Moreover, (M, d) is a complete metric space so that bounded and closed subsets

are compact. From the completeness of the Riemannian manifold M , the exponential map $\exp_p: T_p M \rightarrow M$ is defined by $\exp_p v = \gamma_v(1, p)$, for each $p \in M$.

We denote by R the curvature tensor defined by $R(X, Y) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[Y, X]} Z$, with $X, Y, Z \in \mathcal{X}(M)$, where $[X, Y] = YX - XY$. Moreover, the sectional curvature with respect to X and Y is given by $K(X, Y) = \langle R(X, Y)Y, X \rangle / (\|X\|^2 \|Y\|^2 - \langle X, Y \rangle^2)$, where $\|X\| = \langle X, X \rangle^{1/2}$. If $K(X, Y) \leq 0$ for all X and Y , then M is called a Riemannian manifold of nonpositive curvature and we use the short notation $K \leq 0$.

THEOREM 1.1 *Let M be a complete, simply connected Riemannian manifold with nonpositive sectional curvature. Then M is diffeomorphic to the Euclidean space \mathbb{R}^n , $n = \dim M$. More precisely, at any point $p \in M$, the exponential map \exp_p is a diffeomorphism.*

Proof See Lemma 3.2 of [15, p. 149] or Theorem 4.1 of [46, p. 221]. ■

A complete simply connected Riemannian manifold of nonpositive sectional curvature is called a *Hadamard manifold*. Thus Theorem 1.1 states that if M is a Hadamard manifold, then M has the same topology and differential structure of the Euclidean space \mathbb{R}^n . Furthermore, there are known some similar geometrical properties of the Euclidean space \mathbb{R}^n , such as, given two points there exists an unique geodesic segment that joins them. *In this article, all manifolds M are assumed to be Hadamard and finite dimensional.*

2. Convexity in Hadamard manifolds

In this section, we introduce some fundamental properties and notations of convex analysis on Hadamard manifolds. References of the convex analysis, on the Euclidean space \mathbb{R}^n can be found in [22] and on Riemannian manifolds in [13, 18, 42, 46, 47, 51].

The set $\Omega \subset M$ is said to be *convex* if for any geodesic segment, with end points in Ω , is contained in Ω . Let $\Omega \subset M$ be an open convex set. A function $f: M \rightarrow \mathbb{R}$ is said to be *convex* (respectively, *strictly convex*) on Ω if for any geodesic segment $\gamma: [a, b] \rightarrow \Omega$ the composition $f \circ \gamma: [a, b] \rightarrow \mathbb{R}$ is convex (respectively, strictly convex). Moreover, a function $f: M \rightarrow \mathbb{R}$ is said to be *strongly convex* on Ω with constant $L > 0$ if, for any geodesic segment $\gamma: [a, b] \rightarrow \Omega$, the composition $f \circ \gamma: [a, b] \rightarrow \mathbb{R}$ is strongly convex with constant $L \|\gamma'(0)\|^2$. Take $p \in M$. A vector $s \in T_p M$ is said to be a *subgradient* of f at p , if:

$$f(q) \geq f(p) + \langle s, \exp_p^{-1} q \rangle,$$

for any $q \in M$. The set of all subgradients of f at p , $\partial f(p)$, is called the *subdifferential* of f at p .

The following result provides a characterization of convexity in the case of differentiable functions.

PROPOSITION 2.1 *Let $\Omega \subset M$ be an open convex set and $f: M \rightarrow \mathbb{R}$ a differentiable function on Ω . We say that f is convex on Ω if, and only if, for any $p \in \Omega$:*

$$f(q) - f(p) \geq \left\langle \text{grad} f(p), \exp_p^{-1} q \right\rangle,$$

for all $q \in \Omega$.

Proof See Theorem 5.1 of [51, p. 78]. ■

The most important consequence of the previous proposition is that with f being convex, any of its critical points are global minimum points. Moreover, $0 \in \partial f(p)$ if, and only if, p is a minimum point of f in M , see, for example, [51].

Definition 2.1 Let $\Omega \subset M$ be an open convex set and X a vector field defined in M . X is said to be monotone on Ω , if:

$$\left\langle \exp_q^{-1} p, P_{qp}^{-1} X(p) - X(q) \right\rangle \geq 0, \quad p, q \in \Omega, \quad (4)$$

where P_{qp} is the parallel transport along the geodesic joining q to p . If (4) is satisfied with strict inequality for all $p, q \in \Omega, p \neq q$, then X is said to be strictly monotone. Moreover, X is strongly monotone if there exists $\lambda > 0$ such that:

$$\left\langle \exp_q^{-1} p, P_{qp}^{-1} X(p) - X(q) \right\rangle \geq \lambda d^2(p, q) \quad p, q \in \Omega. \quad (5)$$

The monotone vector fields were first defined in [34]. More results and properties of monotone vector fields can be found in [23,35–37].

Remark 2.1 In the particular case that $M = \mathbb{R}^n$ with the usual metric, inequality (4) and (5) becomes, respectively:

$$\langle p - q, X(p) - X(q) \rangle \geq 0, \quad \langle p - q, X(p) - X(q) \rangle \geq \lambda \|p - q\|^2,$$

because $\exp_q^{-1} p = p - q$ and $P_{qp}^{-1} = I$. Therefore the Definition 2.1 extends the concept of monotone operators from \mathbb{R}^n to Riemannian manifolds.

Now we present an important example of strong monotone vector field being particularly useful in the remainder of this work.

Take $p \in M$ and let $\exp_p^{-1} : M \rightarrow T_p M$ be the inverse of the exponential map. Note that $d(q, p) = \|\exp_p^{-1} q\|$, the map $d^2(\cdot, p) : M \rightarrow \mathbb{R}$ is C^∞ and

$$\text{grad} \frac{1}{2} d^2(q, p) = -\exp_q^{-1} p,$$

(M is a Hadamard manifold), see, for example, Proposition 4.8 of [46, p. 108].

PROPOSITION 2.2 Take $p \in M$. The gradient vector field $\text{grad}(d^2(\cdot, p)/2)$ is strongly monotone with $\lambda = 1$.

Proof See [13]. ■

PROPOSITION 2.3 Let $\Omega \subset M$ be an open convex set and $f : M \rightarrow \mathbb{R}$ a differentiable function on Ω .

- (i) f is convex on Ω if and only if $\text{grad} f$ is monotone on Ω ;
- (ii) f is strictly convex on Ω if and only if $\text{grad} f$ is strictly monotone on Ω ;
- (iii) f is strongly convex on Ω if and only if $\text{grad} f$ is strongly monotone on Ω .

Proof See [13,37,51]. ■

Remark 2.2 Take $p \in M$. From Propositions 2.2 and 2.3 it follows that the map $d^2(\cdot, p)/2$ is strongly convex.

PROPOSITION 2.4 *Let $\Omega \subset M$ be a convex set and $T \subset \mathbb{R}$ a compact set. Let $\psi: M \times T \rightarrow \mathbb{R}$ be a continuous function on $\Omega \times T$ such that $\psi_\tau := \psi(\cdot, \tau): M \rightarrow \mathbb{R}$ is strongly convex on Ω with constant $L > 0$ for all $\tau \in T$. Then, $\phi: M \rightarrow \mathbb{R}$ defined by*

$$\phi(p) := \max_{\tau \in T} \psi(p, \tau)$$

is strongly convex on Ω with constant L . In particular, if ψ_τ is convex for all $\tau \in T$ then ϕ is convex on Ω .

Proof Since T is compact and ψ is continuous, the function ϕ is well defined. Let $\gamma: [a, b] \rightarrow \Omega$ be a geodesic segment. Because ψ_τ is strongly convex with constant L for each $\tau \in T$, we have:

$$(\psi_\tau \circ \gamma)(\alpha t_1 + (1 - \alpha)t_2) \leq \alpha(\psi_\tau \circ \gamma)(t_1) + (1 - \alpha)(\psi_\tau \circ \gamma)(t_2) - \frac{1}{2}(1 - \alpha)\alpha L \|\gamma'(0)\|^2,$$

for all $t_1, t_2 \in [a, b]$ and $\alpha \in [0, 1]$. Thus, taking the maximum in τ on both sides of the above inequality, we obtain:

$$(\phi \circ \gamma)(\alpha t_1 + (1 - \alpha)t_2) \leq \alpha(\phi \circ \gamma)(t_1) + (1 - \alpha)(\phi \circ \gamma)(t_2) - \frac{1}{2}(1 - \alpha)\alpha L \|\gamma'(0)\|^2,$$

which implies that $\phi \circ \gamma: [a, b] \rightarrow \mathbb{R}$ is strongly convex with constant $L \|\gamma'(0)\|^2$. So, ϕ is strongly convex on Ω with constant L . The proof of the second part is immediate. ■

Definition 2.2 Let $\Omega \subset M$ be an open convex set. A function $f: M \rightarrow \mathbb{R}$ is said to be Lipschitz on Ω if there exists a constant $L := L(\Omega) \geq 0$ such that

$$|f(p) - f(q)| \leq Ld(p, q), \quad p, q \in \Omega. \quad (6)$$

Moreover, if for each $p_0 \in \Omega$ there exists $L(p_0) \geq 0$ and $\delta = \delta(p_0) > 0$ such that inequality (6) holds with $L = L(p_0)$ for all $p, q \in B_\delta(p_0) := \{p \in \Omega: d(p, p_0) < \delta\}$, then f is called locally Lipschitz on Ω .

Definition 2.3 Let $\Omega \subset M$ be an open convex set and $f: M \rightarrow \mathbb{R}$ a continuously differentiable function on Ω . The gradient vector field of f , $\text{grad}f$, is said to be Lipschitz with constant $\Gamma \geq 0$ on Ω whenever:

$$\|\text{grad}f(q) - P_{pq}\text{grad}f(p)\| \leq \Gamma d(p, q), \quad p, q \in \Omega,$$

where P_{pq} is the parallel transport along the geodesic joining p to q .

PROPOSITION 2.5 *Let $\Omega \subset M$ be an open convex set and $f: M \rightarrow \mathbb{R}$ a twice continuously differentiable function on Ω . If $\text{Hess}_p f$ is bounded on Ω , then the gradient vector field $\text{grad}f$ is Lipschitz on Ω .*

Proof The proof is an immediate consequence of the fundamental theorem of calculus for vector fields, see for example, [20]. ■

3. Directional derivatives

In this section, we present some properties of the directional derivative of a convex function defined on a Hadamard manifold, including a characterization of the

directional derivative and of the subdifferential of the maximum of a certain class of convex functions. We also give a definition of the generalized directional derivative of a locally Lipschitz function (not necessarily convex), see Azagra et al. [4], and an important property of the subdifferential of the maximum of continuously differentiable functions.

3.1. Directional derivatives of convex functions

In this section, we present the definition of the directional derivative of a convex function defined on a Hadamard manifold and some properties involving its subdifferential, which allow us to obtain an important property of the subdifferential of the maximum of a certain class of convex functions.

Let $\Omega \subset M$ be an open convex set and $f: M \rightarrow \mathbb{R}$ a convex function on Ω . Take $p \in \Omega$, $v \in T_p M$ and $\delta > 0$ and let $\gamma: [-\delta, \delta] \rightarrow \Omega$ be the geodesic segment such that $\gamma(0) = p$ and $\gamma'(0) = v$. Due to the convexity of $f \circ \gamma: [-\delta, \delta] \rightarrow \mathbb{R}$, the function $q_\gamma: (0, \delta] \rightarrow \mathbb{R}$, given by

$$q_\gamma(t) := \frac{f(\gamma(t)) - f(p)}{t} \tag{7}$$

is nondecreasing. Moreover, since f is locally Lipschitzian, it follows that q_γ is bounded near zero. This leads to the following definition:

Definition 3.1 Let $\Omega \subset M$ be an open convex set and $f: M \rightarrow \mathbb{R}$ a convex function on Ω . Then the directional derivative of f at $p \in \Omega$ in the direction of $v \in T_p M$ is defined by

$$f'(p, v) := \lim_{t \rightarrow 0^+} q_\gamma(t) = \inf_{t > 0} q_\gamma(t), \tag{8}$$

where $\delta > 0$ and $\gamma: [-\delta, \delta] \rightarrow \Omega$ is the geodesic segment such that $\gamma(0) = p$ and $\gamma'(0) = v$.

PROPOSITION 3.1 *Let $\Omega \subset M$ be an open convex set and $f: M \rightarrow \mathbb{R}$ a convex function on Ω . Then, for each fixed $p \in \Omega$, the subdifferential $\partial f(p)$ is convex.*

Proof See Theorem 4.6 of [51, p. 74]. ■

PROPOSITION 3.2 *Let $\Omega \subset M$ be an open convex set and $f: M \rightarrow \mathbb{R}$ a convex function on Ω . Then, for each point fixed $p \in \Omega$, the following statements hold:*

- (i) $f'(p, v) = \max_{s \in \partial f(p)} \langle s, v \rangle$, for all $v \in T_p M$;
- (ii) $\partial f(p) = \{s \in T_p M: f'(p, v) \geq \langle s, v \rangle, v \in T_p M\}$.

Proof See [12]. ■

PROPOSITION 3.3 *Let $T \subset \mathbb{R}$ be a compact set, $\Omega \subset M$ an open convex set and $h: M \times T \rightarrow \mathbb{R}$ a continuous function on $\Omega \times T$ such that $h(\cdot, \tau): M \rightarrow \mathbb{R}$ is convex on Ω for all $\tau \in T$. If $f: M \rightarrow \mathbb{R}$ is given by $f(p) = \max_{\tau \in T} h(p, \tau)$, then f is convex on Ω and*

$$f'(p, v) = \max_{\tau \in T(p)} h'(p, \tau, v), \quad p \in \Omega, \quad v \in T_p M,$$

where $T(p) = \{\tau \in T: f(p) = h(p, \tau)\}$. Moreover, if $h(\cdot, \tau)$ is differentiable on Ω for all $\tau \in T$ and $\text{grad}_p h(p, \cdot)$ is continuous for all $p \in \Omega$, then:

$$\partial f(p) = \text{conv}\{\text{grad}_p h(p, \tau): \tau \in T(p)\}.$$

Proof Since T is compact f is well defined and its convexity follows from Proposition 2.4. Now, take $p \in \Omega$, $v \in T_p M$ and the geodesic segment $\gamma: [-\delta, \delta] \rightarrow \Omega$, $\delta > 0$, such that $\gamma(0) = p$ and $\gamma'(0) = v$. Using that T is compact we have $T(p) \neq \emptyset$. Hence, taking $\tau \in T(p)$ we obtain from definition of f and $T(p)$

$$\frac{f(\gamma(t)) - f(p)}{t} \geq \frac{h(\gamma(t), \tau) - h(p, \tau)}{t}, \quad t \in (0, \delta).$$

Given that f and $h(\cdot, \tau)$ are convex, letting t goes to 0, the above inequality yields:

$$f'(p, v) \geq h'(p, \tau, v), \quad p \in \Omega, v \in T_p M, \tau \in T(p).$$

Therefore,

$$f'(p, v) \geq \sup_{\tau \in T(p)} h'(p, \tau, v), \quad p \in \Omega, v \in T_p M. \quad (9)$$

Now, we are going to prove the equality in the above equation. Let $\{t_k\} \subset (0, \delta)$ such that t_k converges to 0 as k goes to $+\infty$. We define

$$p^k := \gamma(t_k), \quad \tau_k \in T(p^k). \quad (10)$$

The last equality implies that $f(\gamma(t_k)) = h(\gamma(t_k), \tau_k)$. Therefore, as f is convex, combining (8) and the definition of f , we obtain:

$$f'(p, v) \leq \frac{f(\gamma(t_k)) - f(p)}{t_k} \leq \frac{h(\gamma(t_k), \tau_k) - h(p, \tau_k)}{t_k}.$$

Since $\{\tau_k\} \subset T$ and T is compact, we can suppose (taking a subsequence, if necessary) that it converges to $\bar{\tau} \in T$ as k goes to $+\infty$. Thus, letting k goes to $+\infty$ in the latter inequality, we have:

$$f'(p, v) \leq h'(p, \bar{\tau}, v),$$

since h is continuous and $x \mapsto h(x, \tau)$ is convex for each $\tau \in T$. Note that, if $\bar{\tau} \in T(p)$, then the last inequality implies that (9) holds with equality. So, in order to prove the first part it is sufficient to prove that $\bar{\tau} \in T(p)$. First note that using (10) and the definitions of $T(p^k)$ and f , we conclude that:

$$h(p^k, \tau_k) = \max_{\tau \in T} h(p^k, \tau) \geq h(p^k, \tau), \quad \tau \in T.$$

Hence, letting k goes to $+\infty$, we have $h(p, \bar{\tau}) \geq h(p, \tau)$, for all $\tau \in T$, which, combined with the definition of f , gives $f(p) = h(p, \bar{\tau})$, and the first part is completed.

To prove the second part, take $p \in \Omega$ and $\tau \in T(p)$. From Proposition 2.1, the convexity of $h(\cdot, \tau)$ implies:

$$h(q, \tau) \geq h(p, \tau) +_p h(p, \tau), \exp_p^{-1} q), \quad q \in \Omega.$$

Because $\tau \in T(p)$ we have $h(p, \tau) = f(p)$, which along with the definition of f as well as with latter equation yields:

$$f(q) \geq f(p) +_p h(p, \tau), \exp_p^{-1} q).$$

So, $\text{grad}_p h(p, \tau) \in \partial f(p)$. Since Proposition 3.1 implies that $\partial f(p)$ is convex, we conclude that:

$$\text{conv} \{ \text{grad}_p h(p, \tau) : \tau \in T(p) \} \subseteq \partial f(p).$$

We claim that this inclusion holds with equality. Indeed, assume by contradiction that:

$$\exists y \in \partial f(p), \quad y \notin \text{conv} \{ \text{grad}_p h(p, \tau) : \tau \in T(p) \}.$$

Due to the fact that $\text{grad}_p h(p, \cdot)$ is continuous and $T(p)$ is a compact set we conclude that the set

$$\text{conv} \{ \text{grad}_p h(p, \tau) : \tau \in T(p) \}$$

is compact. Thus, by the separation theorem on $T_p M$, there exists $v \in T_p M - \{0\}$ and $a \in \mathbb{R}$ such that

$$\langle y, v \rangle > a > \langle \text{grad}_p h(p, \tau), v \rangle, \quad \forall \tau \in T(p).$$

Since $h'(p, \tau, v) = \langle \text{grad}_p h(p, \tau), v \rangle$, it follows from the latter inequality and the first part of the proposition that

$$\langle y, v \rangle > \max_{\tau \in T(p)} h'(p, \tau, v) = f'(p, v).$$

Since $y \in \partial f(p)$, we obtain a contradiction with Proposition 3.2 (i). This proves our claim and concludes the proof. ■

COROLLARY 3.1 *Let $\Omega \subset M$ be a open convex set and $h_i: M \rightarrow \mathbb{R}$ a differentiable convex function on Ω for $i \in I := \{1, \dots, m\}$. If $h: M \rightarrow \mathbb{R}$ is defined by $h(p) := \max_{i \in I} h_i(p)$, then:*

$$\partial h(p) = \text{conv} \{ \text{grad} h_i : i \in I(p) \} = \left\{ y \in T_p M : y = \sum_{i \in I(p)} \alpha_i \text{grad} h_i(p), \sum_{i \in I(p)} \alpha_i = 1, \alpha_i \geq 0 \right\},$$

where $I(p) := \{i \in I : h(p) = h_i(p)\}$. In particular, p minimizes h on Ω , if and only if, there exist $\alpha_i \geq 0$, $i \in I(p)$, such that:

$$0 = \sum_{i \in I(p)} \alpha_i \text{grad} h_i(p), \quad \sum_{i \in I(p)} \alpha_i = 1.$$

Proof It follows directly from Proposition 3.3. ■

3.2. Directional derivatives of locally Lipschitz functions

In the sequel, we present the definition of generalized directional derivative of a locally Lipschitz function (not necessarily convex) and an important property of the subdifferential of the maximum of continuously differentiable functions.

Definition 3.2 Let $\Omega \subset M$ be an open convex set and $f: M \rightarrow \mathbb{R}$ a locally Lipschitz function on Ω . The generalized directional derivative of f at $p \in \Omega$ in the direction $v \in T_p M$ is defined by

$$f^\circ(p, v) := \limsup_{t \downarrow 0, q \rightarrow p} \frac{f(\exp_q t(D \exp_p)_{\exp_p^{-1} q} v)) - f(q)}{t}, \quad (11)$$

where $(D \exp_p)_{\exp_p^{-1} q}$ denotes the differential of \exp_p at $\exp_p^{-1} q$.

It is worth noting that an equivalent definition has appeared in [4].

Remark 3.1 The generalized directional derivative is well defined. Indeed, let $L_p > 0$ be the Lipschitz constant of f in p and $\delta = \delta(p) > 0$ such that

$$|f(\exp_q t(D \exp_p)_{\exp_p^{-1} q} v) - f(q)| \leq L_p d(\exp_q t(D \exp_p)_{\exp_p^{-1} q} v, q), \quad q \in B_\delta(p), \quad t \in [0, \delta).$$

Since $d(\exp_q t(D \exp_p)_{\exp_p^{-1} q} v, q) = t \|(D \exp_p)_{\exp_p^{-1} q} v\|$, the above inequality becomes:

$$|f(\exp_q t(D \exp_p)_{\exp_p^{-1} q} v) - f(q)| \leq L_p t \|(D \exp_p)_{\exp_p^{-1} q} v\|, \quad q \in B_\delta(p), \quad t \in [0, \delta).$$

Since $\lim_{q \rightarrow p} (D \exp_p)_{\exp_p^{-1} q} v = v$, our statement follows from the latter inequality.

Remark 3.2 Note that, if $M = \mathbb{R}^n$ then $\exp_p w = p + w$ and

$$D(\exp_p)_{\exp_p^{-1} q} v = v.$$

In this case, (11) becomes:

$$f_E^\circ(p, v) = \limsup_{t \downarrow 0, q \rightarrow p} \frac{f(q + tv) - f(q)}{t},$$

which is Clarke's generalized directional derivative in the Euclidean case, see [11]. Therefore, the generalized differential derivative on Hadamard manifolds is a natural extension of Clarke's generalized differential derivative.

Next we generalize the definition of subdifferential for locally Lipschitz functions defined on Hadamard manifolds, see Proposition 3.2 (ii).

Definition 3.3 Let $\Omega \subset M$ be an open convex set and $f: M \rightarrow \mathbb{R}$ a locally Lipschitz function on Ω . The generalized subdifferential of f at $p \in \Omega$, denoted by $\partial^\circ f(p)$, is defined by

$$\partial^\circ f(p) := \{w \in T_p M : f^\circ(p, v) \geq \langle w, v \rangle \text{ for all } v \in T_p M\}.$$

Remark 3.3 If f is convex on Ω , then $f^\circ(p, v) = f'(p, v)$ (respectively, $\partial^\circ f(p) = \partial f(p)$) for all $p \in \Omega$, i.e. the directional derivatives (respectively, subdifferential) for Lipschitz functions are a generalization of the directional derivatives (respectively, subdifferential) for convex functions, see [4, Claim 5.4 in the proof of Theorem 5.3].

Definition 3.4 Let $\Omega \subset M$ be an open convex set and $f: M \rightarrow \mathbb{R}$ a locally Lipschitz function on Ω . A point $p \in \Omega$ is a stationary point of f if $0 \in \partial^\circ f(p)$.

LEMMA 3.1 Let $\Omega \subset M$ be an open set. If $f: M \rightarrow \mathbb{R}$ is locally Lipschitz on Ω and $g: M \rightarrow \mathbb{R}$ is continuously differentiable on Ω , then:

$$(f + g)^\circ(p, v) = f^\circ(p, v) + g'(p, v) \quad p \in \Omega, \quad v \in T_p M. \quad (12)$$

As a consequence,

$$\partial^\circ(f+g)(p) = \partial^\circ f(p) + \text{grad } g(p), \quad p \in \Omega. \quad (13)$$

Proof Using the definition of the generalized directional derivative and simple algebraic manipulations, we obtain:

$$(f+g)^\circ(p, v) = \limsup_{t \downarrow 0, q \rightarrow p} \left[\frac{f(\exp_q t(D \exp_p)_{\exp_p^{-1} q} v) - f(q)}{t} + \frac{g(\exp_q t(D \exp_p)_{\exp_p^{-1} q} v) - g(q)}{t} \right].$$

From the upper limit properties together with the definition of the generalized directional derivative and Remark 3.3, we obtain

$$(f+g)^\circ(p, v) \leq f^\circ(p, v) + g'(p, v). \quad (14)$$

Because $f^\circ(p, v) = ((f+g) + (-g))^\circ(p, v)$, the above inequality implies in particular that

$$f^\circ(p, v) \leq (f+g)^\circ(p, v) + (-g)'(p, v),$$

which is equivalent to

$$(f+g)^\circ(p, v) \geq f^\circ(p, v) + g'(p, v).$$

So, the last inequality together with inequality (14) yields the equality (12).

The proof of the equality (13) is an immediate consequence of the equality (12) and the definition of the generalized subdifferential. ■

It is possible to prove that the next result holds with equality. However, we will prove just the inclusion needed to prove our main result.

PROPOSITION 3.4 *Let $\Omega \subset M$ be an open convex set and $I = \{1, \dots, m\}$. Let $f_i: M \rightarrow \mathbb{R}$ be a continuously differentiable function on Ω for all $i \in I$ and $f: M \rightarrow \mathbb{R}$ defined by*

$$f(p) := \max_{i \in I} f_i(p).$$

Then f is locally Lipschitz on Ω and for each $p \in \Omega$

$$\text{conv} \{ \text{grad} f_i(p) : i \in I(p) \} \subset \partial^\circ f(p),$$

where $I(p) := \{i \in I: f_i(p) = f(p)\}$.

Proof Since f_i is continuously differentiable on Ω we conclude that f_i is Lipschitz locally in Ω , for all $i \in I$. Thus, for each $\tilde{p} \in \Omega$ and $i \in I$, there exists $\delta_i, L_i > 0$ such that

$$|f_i(p) - f_i(q)| \leq L_i d(p, q), \quad p, q \in B_{\delta_i}(\tilde{p}).$$

On the other hand,

$$| \max_{i \in I} f_i(p) - \max_{i \in I} f_i(q) | \leq \max_{i \in I} |f_i(p) - f_i(q)|.$$

Combining the last two equations with the definition of f , we obtain:

$$|f(p) - f(q)| \leq L d(p, q) \quad p, q \in B_\delta(\tilde{p}),$$

where $\delta = \min_{i \in I} \delta_i$ and $L = \max_{i \in I} L_i$, which proves the first part.

In order to prove the second part, take $p \in \Omega$, $u \in \text{conv}\{\text{grad} f_i(p) : i \in I(p)\}$ and $v \in T_p M$. Then, there exist a constant $\alpha_i \geq 0$ for $i \in I(p)$ with $\sum_{i \in I(p)} \alpha_i = 1$ such that

$$u = \sum_{i \in I(p)} \alpha_i \text{grad} f_i(p).$$

Since f_i is differentiable for all $i \in I$, simple algebraic manipulation yields

$$\langle u, v \rangle = \sum_{i \in I(p)} \alpha_i \langle \text{grad} f_i(p), v \rangle = \sum_{i \in I(p)} \alpha_i f'_i(p, v).$$

As f is locally Lipschitz at p , the definitions of f , $I(p)$ and generalized directional derivative imply:

$$f'_i(p, v) \leq f^\circ(p, v),$$

which, together with the latter equation, gives $\langle u, v \rangle \leq f^\circ(p, v)$, and the proof follows from the definition of $\partial^\circ f(p)$. \blacksquare

4. Proximal point method for nonconvex problems

In this section, we present an application of the proximal point method to minimize a real-valued function (not necessarily convex) given by the maximum of a certain class of continuously differentiable functions. Our goal is to prove the following theorem:

THEOREM 4.1 *Let $\Omega \subset M$ be an open convex set, $q \in M$ and $I = \{1, \dots, m\}$. Let $f_i: M \rightarrow \mathbb{R}$ be a continuously differentiable function on Ω and continuous on $\bar{\Omega}$ (closure of Ω), for all $i \in I$, and $f: M \rightarrow \mathbb{R}$ defined by*

$$f(p) := \max_{i \in I} f_i(p).$$

Assume that $-\infty < \inf_{p \in M} f(p)$, $\text{grad} f_i$ is Lipschitz on Ω with constant L_i for each $i \in I$ and

$$S_f(f(q)) = \{p \in M : f(p) \leq f(q)\} \subset \Omega, \quad \inf_{p \in M} f(p) < f(q).$$

Take $0 < \bar{\lambda}$ and a sequence $\{\lambda_k\}$ satisfying $\max_{i \in I} L_i < \lambda_k \leq \bar{\lambda}$ and $\hat{p} \in S_f(f(q))$. Then the proximal point method

$$p^{k+1} := \text{argmin}_{p \in M} \left\{ f(p) + \frac{\lambda_k}{2} d^2(p, p^k) \right\}, \quad k = 0, 1, \dots, \quad (15)$$

with starting point $p^0 = \hat{p}$ is well defined, the generated sequence $\{p^k\}$ rests in $S_f(f(q))$ and satisfies only one of the following statements:

- (i) $\{p^k\}$ is finite, i.e., $p^{k+1} = p^k$ for some k and, in this case, p^k is a stationary point of f ;
- (ii) $\{p^k\}$ is infinite and, in this case, any accumulation point of $\{p^k\}$ is a stationary point of f .

Moreover, assume that the minimizer set of f is nonempty, i.e.,

$$(H1) \quad U^* = \{p : f(p) = \inf_{p \in M} f(p)\} \neq \emptyset.$$

Let $c \in (\inf_{p \in M} f(p), f(q))$. If, in addition, the following assumptions hold:

$$(H2) \quad S_f(c) := \{p \in M : f(p) \leq c\} \text{ is convex and } f \text{ is convex on } S_f(c);$$

(H3) For all $p \in S_f(f(q)) \setminus S_f(c)$ and $y(p) \in \partial^\circ f(p)$ we have $\|y(p)\| > \delta > 0$, then the sequence $\{p^k\}$ generated by (15) with

$$\max_{i \in I} L_i < \lambda_k \leq \bar{\lambda}, \quad k = 0, 1, \dots \quad (16)$$

converges to a point $p^* \in U^*$.

Remark 4.1 The continuity of each function f_i on $\bar{\Omega}$ in (H2) guarantees that the level sets of the function f , in particular the solution set U^* , are closed in the topology of the manifold.

In the next remark we show that if Ω is bounded and f_i is convex on Ω , for all $i \in I$, then f satisfies the assumptions (H2) and (H3).

Remark 4.2 If f_i is also a convex function on Ω for each $i \in I$ then by Proposition 2.4, the function f is convex on Ω and the assumption (H2) is satisfied for all $c \leq f(q)$. Moreover, from Remark 3.3,

$$\partial f^\circ(p) = \partial f(p), \quad \forall p \in \Omega. \quad (17)$$

Take $c \in (\inf_{p \in M} f(p), f(q))$ and let us suppose that (H1) hold and Ω is a bounded set. Then, we have

$$0 < \sup\{d(p^*, p) : p^* \in U^*, p \in S_f(f(q)) \setminus S_f(c)\} = \epsilon < +\infty. \quad (18)$$

Let $p^* \in U^*$ be fixed, $p \in S_f(f(q)) \setminus S_f(c)$ and $y(p) \in \partial f(p)$. The convexity of f on Ω implies that:

$$\langle y(p), -\exp_p^{-1} p^* \rangle \geq f(p) - f(p^*).$$

Since $\|y(p)\| \|\exp_p^{-1} p^*\| \geq \langle y(p), -\exp_p^{-1} p^* \rangle$, $d(p^*, p) = \|\exp_p^{-1} p^*\|$, $p \in S_f(f(q)) \setminus S_f(c)$ and U^* is a proper subset of $S_f(c)$, from the above inequality, we obtain:

$$\|y(p)\| d(p^*, p) > c - f(p^*) > 0.$$

Thus, from (18) and the latter inequality

$$\|y(p)\| \epsilon > c - f(p^*) > 0.$$

Therefore, choosing $\delta = (c - f(p^*))/\epsilon$, we have:

$$\|y(p)\| > \delta > 0,$$

which, combined with (17), shows that f satisfies (H3).

In order to prove the above theorem we need some preliminary results. From now on we assume that all assumptions on Theorem 4.1 hold, with the exception of (H1)–(H3), which will be assumed to hold only when explicitly stated.

LEMMA 4.1 For all $\tilde{p} \in M$ and λ satisfying

$$\sup_{i \in I} L_i < \lambda,$$

the function $f_i + (\lambda/2)d^2(\cdot, \tilde{p})$ is strongly convex in Ω with constant $\lambda - \sup_{i \in I} L_i$. Consequently, $f + (\lambda/2)d^2(\cdot, \tilde{p})$ is strongly convex in Ω with constant $\lambda - \sup_{i \in I} L_i$.

Proof Due to the finiteness of I , the function f is well defined. Take $i \in I$, $\tilde{p} \in M$ and define $h_i := f_i + (\lambda/2)d^2(\cdot, \tilde{p})$. Note that $\text{grad } h_i(p) = \text{grad } f_i(p) - \lambda \exp_p^{-1} \tilde{p}$. Thus, for all $p, q \in \Omega$:

$$\begin{aligned} & \langle P_{qp}^{-1} \text{grad } h_i(p) - \text{grad } h_i(q), \exp_q^{-1} p \rangle \\ &= \langle P_{qp}^{-1} \text{grad } f_i(p) - \text{grad } f_i(q), \exp_q^{-1} p \rangle - \lambda \langle P_{qp}^{-1} \exp_p^{-1} \tilde{p} - \exp_q^{-1} \tilde{p}, \exp_q^{-1} p \rangle. \end{aligned}$$

Since $\langle P_{qp}^{-1} \text{grad } f_i(p) - \text{grad } f_i(q), \exp_q^{-1} p \rangle \geq -\|P_{qp}^{-1} \text{grad } f_i(p) - \text{grad } f_i(q)\| \|\exp_q^{-1} p\|$, using equality $d(p, q) = \|\exp_q^{-1} p\|$, Proposition 2.2 and the above equation, we obtain:

$$\langle P_{qp}^{-1} \text{grad } h_i(p) - \text{grad } h_i(q), \exp_q^{-1} p \rangle \geq -\|P_{qp}^{-1} \text{grad } f_i(p) - \text{grad } f_i(q)\| d(q, p) + \lambda d^2(p, q).$$

Now, as $\text{grad } f_i$ is Lipschitz on Ω with constant L_i and the parallel transport is an isometry, the latter equation becomes

$$\langle P_{qp}^{-1} \text{grad } h_i(p) - \text{grad } h_i(q), \exp_q^{-1} p \rangle \geq (\lambda - L_i) d^2(p, q).$$

By hypothesis $\lambda > \sup_{i \in I} L_i$. Hence, the above equation and Definition 2.1 imply that $\text{grad } h_i$ is strongly monotone with constant $\lambda - \sup_{i \in I} L_i$. Therefore, from Proposition 2.3 we conclude that h_i is strongly convex with constant $\lambda - \sup_{i \in I} L_i$. It is easy to see that

$$\max_{i \in I} h_i = f + (\lambda/2) d^2(\cdot, \tilde{p}).$$

Thus using Proposition 2.4 the proposition follows. \blacksquare

COROLLARY 4.1 *The proximal point method (15) applied to f with starting point $p^0 = \hat{p}$ is well defined.*

Proof Assume that $p^k \in S_f(f(q))$ for some k . Note that the minimizers of $\psi_k := f + (\lambda_k/2) d^2(\cdot, p^k)$, in case they exist, are in $L_{\psi_k}(\psi_k(p^k)) \subset S_f(f(q)) \subset \Omega$, more precisely,

$$\text{argmin}_{p \in M} \psi_k(p) = \text{argmin}_{p \in L_{\psi_k}(\psi_k(p^k))} \psi_k(p).$$

As f is continuous on $\bar{\Omega}$, $L_{\psi_k}(\psi_k(p^k))$ is closed in the topology of the manifold M . Moreover, since $\sup_{\tau \in T} L_\tau < \lambda_k$, we conclude from Lemma 4.1 that the application ψ_k is strongly convex on Ω with constant $\beta = \lambda - \sup_{\tau \in T} L_\tau$. In this condition, for ψ_k to have a unique minimizing at M it is sufficient that $L_{\psi_k}(\psi_k(p^k))$ is a bounded set. From the convexity of ψ_k on Ω we have, in particular, that $L_{\psi_k}(\psi_k(p^k))$ is a convex set. Assume by contradiction that $L_{\psi_k}(\psi_k(p^k))$ is unbounded. Then, there exist $v \in T_{p^k} M$ and a geodesic $\gamma : [0, +\infty] \rightarrow L_{\psi_k}(\psi_k(p^k))$ such that $\gamma(0) = p^k$ and $\gamma'(0) = v$. Take $t_0 > 0$. For all $t > t_0$, we have

$$\psi_k(\gamma(t_0)) = \psi_k\left(\gamma\left(\frac{t_0}{t} t + \left(1 - \frac{t_0}{t}\right) 0\right)\right).$$

Now, taking into account that ψ_k is strongly convex with constant β , from the last equality it follows that

$$\psi_k(\gamma(t_0)) \leq \frac{t_0}{t} \psi_k(\gamma(t)) + \left(1 - \frac{t_0}{t}\right) \psi_k(\gamma(0)) - \frac{1}{2} \left(1 - \frac{t_0}{t}\right) \frac{t_0}{t} \beta \|v\|^2 t^2.$$

Because, $\psi_k(\gamma(t)), \psi_k(\gamma(0)) \in L_{\psi_k}(\psi_k(p^k))$, from the last inequality, we obtain

$$\psi_k(\gamma(t_0)) \leq \psi_k(p^k) - \frac{1}{2}(t - t_0)t_0\beta\|v\|^2,$$

which is a contradiction, since $\psi_k(\gamma(t_0))$ is finite and $\psi_k(p^k) - \frac{1}{2}(t - t_0)t_0\beta\|v\|^2$ goes to $-\infty$ as t goes to $+\infty$. Therefore, $L_{\psi_k}(\psi_k(p^k))$ is bounded and, consequently, p^{k+1} is well defined. Since $p^0 = \hat{p} \in S_f(f(q))$, the proof follows from a simple induction argument. ■

LEMMA 4.2 *Let $\{p^k\}$ be the sequence generated by the proximal point method (15). Then the following statements hold:*

- (i) $f(p^{k+1}) + (\lambda_k/2)d^2(p^{k+1}, p^k) \leq f(p^k)$, $k = 0, 1, \dots$;
- (ii) $\{p^k\} \subset S_f(f(q))$;
- (iii) $0 \in \partial\left(f + \frac{\lambda_k}{2}d^2(\cdot, p^k)\right)(p^{k+1})$, $k = 0, 1, \dots$;
- (iv) $-\infty < f = \lim_{k \rightarrow \infty} f(p^k)$;
- (v) $\lim_{k \rightarrow \infty} d(p^{k+1}, p^k) = 0$.

Proof The first item is an immediate consequence of (15), which implies that $\{f(p^k)\}$ is monotonous and nonincreasing from where follows item (ii). Since $\max_{i \in I} L_i < \lambda_k$, Lemma 4.1 implies $f + (\lambda_k/2)d^2(\cdot, p^k)$ convex on Ω , which, together with (15), prove item (iii). Using that $\{f(p^k)\}$ is monotonous nonincreasing and that $-\infty < \inf_{p \in M} f(p)$, item (iv) follows. Finally, item (v) is a consequence of items (i) and (iv). ■

LEMMA 4.3 *Let $\{p^k\}$ be the sequence generated by the proximal point method (15) with λ_k satisfying (16). Assume that (H1) and (H2) holds. If $p^k \in S_f(c)$ for some k then $\{p^k\}$ converges to a point $p^* \in U^* \subset \Omega$.*

Proof By hypotheses, $p^k \in S_f(c)$ for some k , i.e. there exists k_0 such that $f(p^{k_0}) \leq c$. Then, from Lemma 4.2 (i), $\{p^k\} \subset S_f(c)$ for all $k \geq k_0$. On the other hand, from (16) we have

$$\sum_{k=0}^{+\infty} \frac{1}{\lambda_k} = +\infty.$$

Therefore, using (H1) and (H2) the result follows from arguments similar to the ones used in the proof of Theorem 6.1 of [17]. ■

LEMMA 4.4 *Let $\{p^k\}$ be the sequence generated by the proximal point method (15) with λ_k satisfying (16). If (H3) holds then after a finite number of steps the proximal iterates go into the set $S_f(c)$.*

Proof First note that since $\inf_{p \in M} f(p) < c$ we have $S_f(c) \neq \emptyset$. Suppose by contradiction that $p^k \in S_f(f(q)) \setminus S_f(c)$ for all k . From (16) we have $\max_{i \in I} L_i < \lambda_k$. Since $(1/2)d^2(\cdot, p)$ is a differentiable function with

$$\text{grad}((1/2)d^2(q, p)) = -\exp_p^{-1} p,$$

applying Lemma 4.1 with $\lambda = \lambda_k$, Lemma 3.1 with $g = (1/2)d^2(\cdot, p)$ and the first part of Proposition 3.4, we obtain:

$$\partial\left(f + \frac{\lambda_k}{2}d^2(\cdot, p^k)\right)(p) = \partial^\circ f(p) - \lambda_k \exp_p^{-1} p^k, \quad k = 0, 1, \dots$$

Hence, it is easily concluded from the last equality and Lemma 4.2 (iii) that:

$$\lambda_k \exp_{p^{k+1}}^{-1} p^k \in \partial^\circ f(p^{k+1}), \quad k = 0, 1, \dots$$

Since $p^{k+1} \in S_f(f(q)) \setminus S_f(c)$, assumption (H3) and the latter equation give

$$\|\lambda_k \exp_{p^{k+1}}^{-1} p^k\| > \delta, \quad k = 0, 1, \dots$$

From (16) we have $\lambda_k \leq \bar{\lambda}$. As $d(p^k, p^{k+1}) = \|\exp_{p^{k+1}}^{-1} p^k\|$ the last inequality implies that

$$d(p^k, p^{k+1}) > \frac{\delta}{\bar{\lambda}}, \quad k = 0, 1, \dots$$

So, from Lemma 4.2 (v) we arrive at a contradiction. Therefore there exists k_0 such that $f(p^{k_0}) \leq c$ and the result follows from Lemma 4.2 (i). ■

Proof of Theorem 4.1

Proof The well definedness of the proximal point method follows from Corollary 4.1. Let $\{p^k\}$ be the sequence generated by the proximal point method. As $p^0 = \hat{p} \in S_f(f(q))$, Lemma 4.2 (i) implies that the whole sequence lies in $S_f(f(q))$. From Lemma 4.2 (iii), we have:

$$0 \in \partial \left(f + \frac{\lambda_k}{2} d^2(\cdot, p^k) \right) (p^{k+1}), \quad k = 0, 1, \dots$$

Since $\max_{i \in I} L_i < \lambda_k$, Lemma 4.1 implies that $f_i + (\lambda_k/2)d^2(\cdot, p^k)$ and $f + (\lambda_k/2)d^2(\cdot, p^k)$ are strongly convex. Thus, applying Corollary 3.1 with $h_i = f_i + (\lambda_k/2)d^2(\cdot, p^k)$ and $h = f + (\lambda_k/2)d^2(\cdot, p^k)$ we conclude that there exists constant $\alpha_i^{k+1} \geq 0$ with $i \in I(p^{k+1})$ such that

$$0 = \sum_{i \in I(p^{k+1})} \alpha_i^{k+1} \text{grad} \left(f_i + \frac{\lambda_k}{2} d^2(\cdot, p^k) \right) (p^{k+1}), \quad \sum_{i \in I(p^{k+1})} \alpha_i^{k+1} = 1.$$

This tells us that

$$0 = \sum_{i \in I(p^{k+1})} \alpha_i^{k+1} \text{grad} f_i(p^{k+1}) - \lambda_k \exp_{p^{k+1}}^{-1} p^k, \quad \sum_{i \in I(p^{k+1})} \alpha_i^{k+1} = 1, \quad k = 0, 1, \dots \quad (19)$$

If the sequence $\{p^k\}$ is finite then there exists k such that $p^{k+1} = p^k$. In this case, $\exp_{p^{k+1}}^{-1} p^k = 0$ and the first equality in (19) becomes:

$$0 = \sum_{i \in I(p^{k+1})} \alpha_i^{k+1} \text{grad} f_i(p^{k+1}),$$

which, together with Proposition 3.4, implies that $0 \in \partial^\circ f(p^k)$. Hence, p^k is a stationary point of f .

Now, assume that sequence $\{p^k\}$ is infinite and \bar{p} is an accumulation point of it. Let $\{\alpha_i^{k+1}\} \subset \mathbb{R}^m$ be the sequence defined by

$$\alpha^{k+1} = (\alpha_1^{k+1}, \dots, \alpha_m^{k+1}), \quad \alpha_j^{k+1} = 0, \quad j \in I \setminus I(p^{k+1}).$$

Since $\sum_{i \in I(p^{k+1})} \alpha_i^{k+1} = 1$ we have $\|\alpha^{k+1}\|_1 = 1$ for all k , where $\|\cdot\|_1$ denotes the sum norm in \mathbb{R}^m . Thus $\{\alpha^{k+1}\}$ is bounded. Let $\{p^{k_s+1}\}$ and $\{\alpha^{k_s+1}\}$ be the subsequence of $\{p^{k+1}\}$ and $\{\alpha^{k+1}\}$, respectively, such that $\lim_{s \rightarrow +\infty} p^{k_s+1} = \bar{p}$ and $\lim_{s \rightarrow +\infty} \alpha^{k_s+1} = \bar{\alpha}$. As f is continuous on Ω , Lemma 4.2 (ii) implies that $\bar{p} \in S_f(f(q)) \subset \Omega$. Since I is finite we can assume without loss of generality that

$$I(p^{k_1+1}) = I(p^{k_2+1}) = \dots =: \bar{I}, \quad (20)$$

and Equation (19) becomes

$$0 = \sum_{i \in \bar{I}} \alpha_i^{k_s+1} \text{grad} f_i(p^{k_s+1}) - \lambda_{k_s} \exp_{p^{k_s+1}}^{-1} p^{k_s}, \quad \sum_{i \in \bar{I}} \alpha_i^{k_s+1} = 1, \quad s = 1, 2, \dots \quad (21)$$

On the other hand, Lemma 4.2 (v) implies

$$\lim_{s \rightarrow \infty} d(p^{k_s+1}, p^{k_s}) = \lim_{s \rightarrow \infty} \|\exp_{p^{k_s+1}}^{-1} p^{k_s}\| = 0.$$

As λ_{k_s} is bounded, $\lim_{s \rightarrow +\infty} p^{k_s+1} = \bar{p}$ and $\lim_{s \rightarrow +\infty} \alpha^{k_s+1} = \bar{\alpha}$, letting s go to $+\infty$ in the above equality, from (21) we conclude

$$0 = \sum_{i \in \bar{I}} \bar{\alpha}_i \text{grad} f_i(\bar{p}), \quad \sum_{i \in \bar{I}} \bar{\alpha}_i = 1.$$

Using definition of $I(\bar{p})$, Equation (20) and the continuity of f , we obtain $\bar{I} \subset I(\bar{p})$. Therefore, as $\bar{p} \in \Omega$, it follows from Proposition 3.4 that

$$0 \in \partial^\circ f(\bar{p}),$$

i.e., \bar{p} is a stationary point of f , which concludes the proof of the first part of the theorem.

The second part follows from Lemmas 4.3 and 4.4. ■

5. Examples

In this section we present two examples. In the first example, we consider a nonconvex minimization problem where the objective function is defined on a Hadamard manifold with curvature identically zero. In the next example, we ‘generalize’ this example to one where the curvature of the Hadamard manifold is not identically zero. In both the examples, the classical local proximal point method ([24]) as well as the Riemannian proximal point method ([18]) does not apply. However, the method proposed in this article applies.

5.1. Example 1

Let $(\mathbb{R}_{++}, \langle \cdot, \cdot \rangle)$ be the Riemannian manifold, where $\mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$ and $\langle \cdot, \cdot \rangle$ is the Riemannian metric $\langle u, v \rangle = g(x)uv$ with $g: \mathbb{R}_{++} \rightarrow (0, +\infty)$. The Christoffel symbol and the geodesic equation are given by

$$\Gamma(x) = \frac{1}{2} g^{-1}(x) \frac{dg(x)}{dx} = \frac{d}{dx} \ln \sqrt{g(x)}, \quad \frac{d^2 x}{dt^2} + \Gamma(x) \left(\frac{dx}{dt} \right)^2 = 0,$$

respectively. Moreover, in relation to the twice differentiable function $h: \mathbb{R}_{++} \rightarrow \mathbb{R}$, the gradient and the Hessian of h are given by

$$\text{grad } h = g^{-1}h', \quad \text{hess } h = h'' - \Gamma h',$$

respectively, where h' and h'' denote the first and second derivatives of h in the Euclidean sense. For more details see [48]. In the particular case of $g(x) = x^{-2}$,

$$\Gamma(x) = -x^{-1}, \quad \text{grad } h(x) = x^2 h'(x), \quad \text{hess } h(x) = h''(x) + x^{-1} h'(x). \quad (22)$$

Moreover, the map $\varphi: \mathbb{R} \rightarrow \mathbb{R}_{++}$ defined by $\varphi(x) = e^x$ is an isometry between the Euclidean space \mathbb{R} and the manifold $(\mathbb{R}_{++}, \langle, \rangle)$ and the Riemannian distance $d: \mathbb{R}_{++} \times \mathbb{R}_{++} \rightarrow \mathbb{R}_+$ is given by

$$d(x, y) = |\varphi^{-1}(x) - \varphi^{-1}(y)| = \left| \ln \frac{x}{y} \right|, \quad (23)$$

see, for example [14]. Therefore, $(\mathbb{R}_{++}, \langle, \rangle)$ is a Hadamard manifold and the unique geodesic $x: \mathbb{R} \rightarrow \mathbb{R}_{++}$ with initial conditions $x(0) = x_0$ and $x'(0) = v$ is given by

$$x(t) = x_0 e^{(v/x_0)t}.$$

From the above equation it can be easily seen that any interval $I \subset \mathbb{R}_{++}$ is a convex set of the manifold $(\mathbb{R}_{++}, \langle, \rangle)$.

Let $f_1, f_2, f: \mathbb{R}_{++} \rightarrow \mathbb{R}$, respectively, be given by

$$f_1(x) = \ln(x), \quad f_2(x) = -\ln(x) + e^{-2x} - e^{-2}, \quad f(x) = \max_{j=1,2} f_j(x),$$

and consider the problem

$$\begin{aligned} & \min f(x) \\ & \text{s.t. } x \in \mathbb{R}_{++}. \end{aligned} \quad (24)$$

Take a sequence $\{\lambda_k\}$ satisfying $0 < \lambda_k$. From (23), the proximal point method (15) becomes

$$x^{k+1} := \operatorname{argmin}_{x \in \mathbb{R}_{++}} \left\{ f(x) + \frac{\lambda_k}{2} \ln^2 \left(\frac{x}{x^k} \right) \right\}, \quad k = 0, 1, \dots$$

Note that $-\infty < \inf_{x \in \mathbb{R}_{++}} f(x) = 0$ and, f_1 and f_2 being twice differentiable functions on \mathbb{R}_{++} , the last expression in (22) implies that

$$\text{hess } f_1(x) = 0 \quad \text{and} \quad \text{hess } f_2(x) = \left(4 - \frac{2}{x} \right) e^{-2x}, \quad x \in \mathbb{R}_{++}. \quad (25)$$

Let $0 < \epsilon < 1/4$, $q = 5/16$ and $\Omega = (\epsilon, +\infty)$. So, $0 = \inf_{x \in \mathbb{R}_{++}} f(x) < f(q)$ and $S_f(f(q)) \subset \Omega$. Moreover, $\text{hess } f_1, \text{hess } f_2$ are bounded on Ω and consequently $\text{grad } f_1, \text{grad } f_2$ are Lipschitz on Ω . We denote by L_i the Lipschitz constant of $\text{grad } f_i$, $i = 1, 2$. Clearly, the assumption (H1) of the Theorem 4.1 is verified with $U^* = \{1\}$.

We claim that there exists $c \in (0, f(1/2)) = (0, f_1(1/2))$ such that $S_f(c)$ is convex and f is convex on $S_f(c)$ (in the Riemannian sense). Indeed, as $\text{hess } f_1 \geq 0$ in \mathbb{R}_{++} and $\text{hess } f_2 \geq 0$ in $[1/2, +\infty)$, Theorem 6.2 of [14] implies that f_1 is convex on \mathbb{R}_{++} and f_2 is convex on $[1/2, +\infty)$. Thus, it follows from Proposition 2.4 that f is convex

on $[1/2, +\infty)$. Note that for all $c \in (0, f(1/2))$, we have $S_f(c) \cap [1/2, +\infty) = S_f(c)$. Hence, from the convexity of f on $(1/2, +\infty)$ we conclude that $S_f(c)$ is convex, which proves the claim. So, $S_f(c)$ and f satisfy assumption (H2) of Theorem 4.1, for example with $c = f(3/4)$.

Now, note that $S_f(f(q)) \setminus S_f(c) = [5/16, 3/4] \cup (a, b]$, where $a = (4/3)e^{(e^{-3/2} - e^{-2})}$ and $b = (16/5)e^{(e^{-5/8} - e^{-2})}$. Moreover, f is differentiable on $S_f(f(q)) \setminus S_f(c)$ with $\text{grad} f(x) = \text{grad} f_1(x)$ for $x \in [a, b]$ and $\text{grad} f(x) = \text{grad} f_2(x)$ for $x \in [5/16, 3/4]$. From the second expression in (22), we have:

$$\text{grad} f(x) = x, \quad x \in (a, b] \quad \text{and} \quad \text{grad} f(x) = -x - 2x^2 e^{-2x}, \quad x \in [5/16, 3/4].$$

Thus, we have $\|\text{grad} f(x)\| \geq \|\text{grad} f(5/16)\| > 2/5$ and f satisfies the assumption (H3) of Theorem 4.1.

Summarizing, all assumptions of Theorem 4.1 are satisfied with $\Omega = (\epsilon, +\infty)$, $q = 5/16$, $c = f(3/4)$ and $\delta = 2/5$. Therefore, letting $x^0 \in \mathbb{R}_{++}$ and $\bar{\lambda} > 0$ such that $x^0 \in S_f(f(q))$ and $\max_{i \in I} L_i < \lambda_k \leq \bar{\lambda}$, the proximal point method may be applied for solving the above nonconvex problem.

Remark 5.1 The function $f(x) = \max\{\ln(x), -\ln(x) + e^{-2x} - e^{-2}\}$, in the above example, is nonconvex (in the Euclidean sense) when restricted to any open neighbourhood containing its minimizer $x^* = 1$. Therefore, the classical local proximal point method [24] cannot be applied to minimize this function. Also, as f is nonconvex in the Riemannian sense as well, the Riemannian proximal point method [18] cannot be applied to minimize this function either.

5.2. Example 2

Let \mathbb{S}^n be the set of the symmetric matrices, \mathbb{S}_+^n be the cone of the symmetric positive semi-definite matrices and \mathbb{S}_{++}^n be the cone of the symmetric positive-definite matrices both $n \times n$. For $X, Y \in \mathbb{S}_+^n$, $Y \succeq X$ (or $X \preceq Y$) means that $Y - X \in \mathbb{S}_+^n$ and $Y \succ X$ (or $X \prec Y$) means that $Y - X \in \mathbb{S}_{++}^n$. We will denote the Frobenius norm by $\|\cdot\|_F$.

Following Rothaus [45], let $M := (\mathbb{S}_{++}^n, \langle \cdot, \cdot \rangle)$ be the Riemannian manifold endowed with the Riemannian metric induced by the Euclidean Hessian of $\Psi(X) = -\ln \det X$,

$$\langle U, V \rangle = \text{tr}(V \Psi''(X) U) = \text{tr}(V X^{-1} U X^{-1}), \quad X \in M, \quad U, V \in T_X M, \quad (26)$$

where $\text{tr}(A)$ denotes the trace of matrix $A \in \mathbb{S}^n$ and $T_X M \approx \mathbb{S}^n$, with the corresponding norm denoted by $\|\cdot\|$. In this case the unique geodesic segment connecting any $X, Y \in M$ is given by

$$\gamma(t) = X^{1/2} (X^{-1/2} Y X^{-1/2})^t X^{1/2}, \quad t \in [0, 1],$$

see, for instance, [39]. More precisely, M is a Hadamard manifold, see, for example, [25, Theorem 1.2. p. 325]. From the above equality it is immediate that

$$\gamma'(0) = X^{1/2} \ln(X^{-1/2} Y X^{-1/2}) X^{1/2}.$$

Thus, for each $X \in M$, $\exp_X^{-1}: M \rightarrow T_X M$ and $\exp_X: T_X M \rightarrow M$ are given, respectively, by

$$\exp_X^{-1} Y = X^{1/2} \ln(X^{-1/2} Y X^{-1/2}) X^{1/2}, \quad \exp_X V = X^{1/2} e^{(X^{-1/2} V X^{-1/2})} X^{1/2}. \quad (27)$$

Now, since the Riemannian distance d is given by $d(X, Y) = \|\exp_X^{-1} Y\|$, from (26) along with first expression in (27), we conclude that:

$$d^2(X, Y) = \text{tr}(\ln^2 X^{-1/2} Y X^{-1/2}) = \sum_{i=1}^n \ln^2 \lambda_i(X^{-1/2} Y X^{-1/2}), \quad (28)$$

where $\lambda_i(X^{-1/2} Y X^{-1/2})$ denotes the i th eigenvalue of the matrix $X^{-1/2} Y X^{-1/2}$. The gradient and the Hessian of a twice differentiable function $F: \mathbb{S}_{++}^n \rightarrow \mathbb{R}$ is given, respectively, by

$$\text{grad} F(X) = X F'(X) X, \quad \text{hess} F(X)(V, V) = \text{tr}(V F''(X) V) + \text{tr}(F'(X) V X^{-1} V), \quad V \in T_X M,$$

where $F'(X)$ and $F''(X)$ are the Euclidean gradient and Hessian, respectively. We remind that a twice differentiable function $F: \mathbb{S}_{++}^n \rightarrow \mathbb{R}$ is convex on the manifold M if it satisfies the condition:

$$\text{hess} F(X)(V, V) \geq 0, \quad X \in M, \quad V \in T_X M.$$

Let $F_1, F_2, F_3: \mathbb{S}_{++}^n \rightarrow \mathbb{R}$ be given, respectively, by

$$F_1(X) = \ln \det X, \quad F_2(X) = -4 \ln \det X + e^{-2\text{tr} X} - e^{-2n}, \quad F_3(X) = \text{tr} X^{-1} - n.$$

Note that F_1, F_2 and F_3 are twice differentiable functions on \mathbb{S}_{++}^n . Taking $X \in T_X M$ and $V \in T_X M$, the Euclidean gradients are given by

$$F_1'(X) = X^{-1}, \quad F_2'(X) = -4X^{-1} - 2e^{-2\text{tr} X}, \quad F_3'(X) = -X^{-2},$$

and the Euclidean Hessians are given by

$$\begin{aligned} F_1''(X)V &= -X^{-1} V X^{-1}, \\ F_2''(X)V &= 4X^{-1} V X^{-1} + 4e^{-2\text{tr} X} V, \\ F_3''(X)V &= X^{-1} V X^{-2} + X^{-2} V X^{-1}. \end{aligned}$$

Thus, the Riemannian gradients of F_1, F_2 and F_3 are given by

$$\text{grad} F_1(X) = X, \quad \text{grad} F_2(X) = -4X - 2e^{-2\text{tr} X} X^2, \quad \text{grad} F_3(X) = -I, \quad (29)$$

where I denotes the identity matrix, and Riemannian Hessians of F_1, F_2 and F_3 are given by

$$\begin{aligned} \text{hess} F_1(X)(V, V) &= 0, \\ \text{hess} F_2(X)(V, V) &= 2e^{-2\text{tr} X} \text{tr}[(2I - X^{-1})V^2], \\ \text{hess} F_3(X)(V, V) &= \text{tr}(X^{-1} V X^{-2} V) = \|X^{-1} V X^{-1/2}\|_F^2. \end{aligned} \quad (30)$$

It can be easily seen that the functions F_1 and F_3 are convex everywhere and the function F_2 is convex on any convex subset of the set

$$C := \{X \in \mathbb{S}_{++}^n : 2I - X^{-1} \succ 0\} = \{X \in \mathbb{S}_{++}^n : \lambda_{\min}(X) > 1/2\}, \quad (31)$$

where $\lambda_{\min}(A)$ denotes the minimum eigenvalue of the matrix A .

Let $F: \mathbb{S}_{++}^n \rightarrow \mathbb{R}$ be given by

$$F(X) = \max_{j=1,2,3} F_j(X).$$

From Proposition 2.4 F is convex on any convex subset of C . Consider the following optimization problem:

$$\begin{aligned} & \min F(X) \\ & \text{s.t. } X \in \mathbb{S}_{++}^n. \end{aligned} \tag{32}$$

If $\{\eta_k\}$ is a sequence satisfying $0 < \eta_k$, then from (28) the proximal point method (15) becomes

$$X_{k+1} = \arg \min_{X \in \mathbb{S}_{++}^n} \left\{ f(X) + \frac{\eta_k}{2} \sum_{i=1}^n \ln^2 \lambda_i \left(X^{-\frac{1}{2}} X_k X^{-\frac{1}{2}} \right) \right\}.$$

Define the sets U_1 , U_2 and U_3 as

$$\begin{aligned} U_1 &:= \{X \in \mathbb{S}_{++}^n : \det X = 1\}, \\ U_2 &:= \{X \in \mathbb{S}_{++}^n : \det X > 1\}, \\ U_3 &:= \{X \in \mathbb{S}_{++}^n : \det X < 1\}, \end{aligned}$$

and note that $\mathbb{S}_{++}^n = U_1 \cup U_2 \cup U_3$ and $U_i \cap U_j = \emptyset$, for $i, j = 1, 2, 3$, $i \neq j$.

Claim 5.1

$$F(x) = \begin{cases} F_3(X) \geq 0, & X \in U_1, \\ \max\{F_1(X), F_3(X)\} > 0, & X \in U_2, \\ \max\{F_2(X), F_3(X)\} > 0, & X \in U_3. \end{cases} \tag{33}$$

In particular, $\inf_{\mathbb{S}_{++}^n} F(X) = 0$ and $U^* = \{I\}$, where U^* is the solution set of the problem (32) and I is the $n \times n$ identity matrix.

Proof First of all, note that for each $X \in \mathbb{S}_{++}^n$, from the relation between arithmetic–geometric mean, we have

$$\sqrt[n]{\det X} \leq \frac{\text{tr } X}{n}. \tag{34}$$

If $X \in U_1$ then $\det X = 1$. Using the above inequality we obtain $\text{tr } X \geq n$ which implies

$$F_2(X) = e^{-2\text{tr } X} - e^{-2n} \leq 0 = F_1(X).$$

On the other hand, $\det X^{-1} = 1$. Hence, (34) gives us $\text{tr } X^{-1} - n \geq 0$ and thus $F_3(X) \geq 0$. So, from the definition of F and taking into account the above inequality, we conclude

$$F(X) = F_3(X) \geq 0, \quad X \in U_1. \tag{35}$$

If $X \in U_2$ then $\det X > 1$. Thus, from (34) we also obtain $\text{tr } X > n$. Hence,

$$F_2(X) = -4 \ln \det X + e^{-2\text{tr } X} - e^{-2n} < -\ln \det X < 0.$$

Since, $\det X > 1$ we have $F_1(X) = \ln \det X > 0$. Therefore,

$$F(X) = \max\{F_1(X), F_3(X)\} > 0, \quad X \in U_2. \tag{36}$$

Finally, if $X \in U_3$ then $\det X^{-1} > 1$. Hence, inequality (34) implies that $\text{tr} X^{-1} > n$ and consequently $F_3(X) = \text{tr} X^{-1} - n > 0$. As $\det X < 1$ we have $F_1(X) = \ln \det(X) < 0$. Thus,

$$F(X) = \max\{F_2(X), F_3(X)\} > 0, \quad X \in U_3, \quad (37)$$

and the first part of the claim is proved.

From the first part we conclude that $F(X) \geq 0$, for all $X \in \mathbb{S}_{++}^n$. Since $F(I) = 0$ we have $\inf_{\mathbb{S}_{++}^n} F(X) = 0$. In order to prove the last statement, note that if $F(X) = 0$ then $X \in U_1$. Definition of U_1 and (33) give us $\det X^{-1} = 1$ and $F(X) = \text{tr}(X^{-1}) - n$ for all $X \in U_1$. Hence, if $X \in U^*$ then $\text{tr}(X^{-1}) = n$ and $\det X^{-1} = 1$, which implies that (34) holds with equality. On the other hand, (34) holds with equality if only if $\lambda_1(X) = \dots = \lambda_n(X) > 0$. As $\text{tr}(X^{-1}) = n$ we conclude that $\lambda_i(X) = 1$ for $i = 1, \dots, n$ and the result follows. ■

Claim 5.2 $F_1(X) > F_3(X)$ for $X \in A$, where

$$A := \{X \in \mathbb{S}_{++}^n : X - I \succ 0\} = \{X \in \mathbb{S}_{++}^n : \lambda_{\min}(X) > 1\} \subset U_2.$$

Consequently, $F(X) = F_1(X)$ for $X \in A$.

Proof Define the following function

$$\psi(X) := F_1(X) - F_3(X).$$

From the mean value theorem on the Euclidean space \mathbb{S}_{++}^n there exists $\tilde{X} \in U_2$ such that

$$\psi(X) = \psi(I) + \text{tr}(\psi'(\tilde{X})(X - I)) = \text{tr}((\tilde{X}^{-1} + \tilde{X}^{-2})(X - I)).$$

If $X \in A$ then $X - I \in \mathbb{S}_{++}^n$. Since $X - I$ and $\tilde{X}^{-1} + \tilde{X}^{-2}$ belong to \mathbb{S}_{++}^n and the trace of the product of positive-definite matrices is positive, we conclude from the above equality that $\psi(X) = F_1(X) - F_3(X) > 0$ for all $X \in A$, which proves the first part of the claim. The second part of the claim follows from Claim 5.1. ■

Claim 5.3 $F(X) = F_2(X)$ for $X \in B$, where

$$B := \{X \in \mathbb{S}_{++}^n : (1/4)I \prec X \prec I\} = \{X \in \mathbb{S}_{++}^n : 1/4 < \lambda_{\min}(X), \lambda_{\max}(X) < 1\}.$$

Proof Note that $B \subset U_3 \cap \{X \in \mathbb{S}_{++}^n : \text{tr} X \leq n\}$. So, from (37), to prove this claim it suffices to verify that $F_2(X) > F_3(X)$ for all $X \in B$. Let $\phi: \mathbb{R}_{++} \rightarrow \mathbb{R}$ be the function given $\phi(t) := -4 \ln t - 1/t + 1$. From the convexity of $-4 \ln t$, we have

$$\phi(t) \geq -4 \ln 1 - 4(t - 1) - \frac{1}{t} + 1 > 0, \quad t \in (1/4, 1). \quad (38)$$

On the other hand, the definitions of F_2 , F_4 and ϕ yield:

$$F_2(X) - F_4(X) = e^{-2\text{tr} X} - e^{-2n} + \sum_{j=1}^n \phi(\lambda_j(X)).$$

Since $e^{-2\text{tr} X} - e^{-2n} > 0$ for $X \in B$, combining (38) with the latter equality we conclude that $F_2(X) > F_3(X)$ for $X \in B$. Hence, the desired equality follows. ■

It is easy to verify that F is a coercive function. Thus, $S_F(a)$ is bounded for all $a \in \mathbb{R}$, i.e. all the level sets of F are bounded. Take

$$Q := \text{diag}(1/4, \dots, 1/4). \quad (39)$$

As $S_F(F(Q)) \subset \mathbb{S}_{++}^n$ is a bounded set, then there exists $\kappa > 4$ sufficiently large so that

$$\lambda_{\min}(X) > 1/\kappa, \quad \lambda_{\max}(X) < \kappa, \quad X \in S_F(F(Q)). \quad (40)$$

Now we define

$$\Omega := \{X \in \mathbb{S}_{++}^n : \text{tr } X^{-1} < \kappa n\}.$$

Since the function $\mathbb{S}_{++}^n \ni X \mapsto \text{tr } X^{-1}$ is convex, the set Ω is convex. From Claim 5.1 $0 = \inf_{\mathbb{S}_{++}^n} F(X) < F(Q)$ and, consequently, $\text{int } S_F(F(Q)) \neq \emptyset$, where $\text{int } A$ represents the interior of the set A . Moreover, it is immediate to verify that

$$C \subset \Omega, \quad I \in \text{int } S_F(F(Q)) \subset S_F(F(Q)) \subset \Omega.$$

Claim 5.4 The gradient vector fields $\text{grad } F_1$, $\text{grad } F_2$ and $\text{grad } F_3$ are Lipschitz on Ω .

Proof From the first equality in (30) and Proposition 2.5 it is immediate that the gradient vector field $\text{grad } F_1$ is Lipschitz. Now, the second equality in (30) implies that:

$$|\text{hess } F_2(X)(V, V)| = 2e^{-2\text{tr } X} |\text{tr}[(2I - X^{-1})V^2]|, \quad X \in M, V \in T_X M.$$

As $\text{tr } X > 0$ and using the Cauchy–Schwarz inequality, we obtain from the above equality that:

$$|\text{hess } F_2(X)(V, V)| \leq 2\|2I - X^{-1}\|_F \|V^2\|_F, \quad X \in M, V \in T_X M.$$

Hence, using the definition of the metric we conclude that:

$$|\text{hess } F_2(X)(V, V)| \leq 2(2\sqrt{n} + \|X^{-1}\|_F) \|V^2\|_F, \quad X \in M, V \in T_X M.$$

Thus, as $\|X^{-1}\|_F \leq \text{tr } X^{-1}$, from the latter inequality along with the definition of Ω , we conclude that:

$$|\text{hess } F_2(X)(V, V)| \leq 4\sqrt{n} + 2\kappa n, \quad X \in \Omega, \|V\|_F = 1. \quad (41)$$

On the other hand, using the third equality in (30) and the Cauchy–Schwarz inequality, we obtain:

$$|\text{hess } F_3(X)(V, V)| = |\text{tr}(X^{-1} V X^{-2} V)| \leq \|X^{-1} V X^{-1}\|_F \|V X^{-1}\|_F, \quad X \in M, V \in T_X M.$$

Since $\|\cdot\|_F$ is submultiplicative and reusing that $\|X^{-1}\|_F \leq \text{tr } X^{-1}$ the above inequality implies that:

$$|\text{hess } F_3(X)(V, V)| \leq (\text{tr } X^{-1})^3 \|V\|_F^2, \quad X \in M, V \in T_X M.$$

Thus,

$$|\text{hess } F_3(X)(V, V)| \leq \kappa^3 n^3, \quad X \in \Omega, \|V\|_F = 1. \quad (42)$$

From (41) and (42) we conclude that $\text{hess } F_2$ and $\text{hess } F_3$ are also bounded operators when restricted to Ω . Therefore, from Proposition 2.5, $\text{grad } F_1$, $\text{grad } F_2$ and $\text{grad } F_3$ are Lipschitz on Ω . ■

We denote by L_i the Lipschitz constant of $\text{grad } f_i$, $i = 1, 2, 3$, on Ω .

From Claim 5.1 the assumption (H1) of Theorem 4.1 is satisfied with $U^* = \{I\}$. The next result will be useful to assure that the assumption (H2) of Theorem 4.1 is satisfied.

Claim 5.5 There exists $\sigma > 0$ and $\hat{X} \in \mathbb{S}_{++}^n \setminus \{I\}$ such that,

$$S_F(F(\hat{X})) \subset B_\sigma(I) \subset \text{int } S_F(F(Q)) \cap C,$$

where $B_\sigma(I) := \{X \in \mathbb{S}_{++}^n : d(X, I) < \sigma\}$, C is defined in (31) and Q in (39).

Proof Since $U^* = \{I\}$ and $I \neq Q$, we conclude that $I \in \text{int } S_F(F(Q)) \cap C$. So, it can be easily seen that there exists $\sigma > 0$ such that:

$$B_\sigma(I) S_F(F(Q)) \cap C,$$

proving the last inclusion. In order to prove the first inclusion, assume by contradiction that for every $X \in B_\sigma(I)$ we have $S_F(F(X)) \cap (\mathbb{S}_{++}^n \setminus B_\sigma(I)) \neq \emptyset$. Therefore, there exist sequences $\{X^k\}, \{Y^k\} \subset \mathbb{S}_{++}^n$ such that

$$X^k \in B_\sigma(I), \quad \lim_{k \rightarrow \infty} X^k = I, \quad Y^k \in S_F(F(X^k)) \setminus B_\sigma(I), \quad k = 0, 1, \dots \quad (43)$$

Since $\{X^k\} \in B_\sigma(I) \subset S_F(F(Q))$ we have $S_F(F(X^k)) \subset S_F(F(Q))$ for $k = 0, 1, \dots$. Hence, $\{Y^k\} \subset S_F(F(Q))$. As $S_F(F(Q))$ is a bounded set, we assume (taking a subsequence, if necessary) that $\{Y^k\}$ converges to some $\bar{Y} \in S_F(F(Q))$. Thus, continuity of F along with the two last equations in (43) imply that:

$$F(\bar{Y}) \leq F(I), \quad \bar{Y} \neq I,$$

and, since $U^* = \{I\}$, we obtain a contradiction, which proves the claim. ■

From the above claim along with the convexity of the ball $B_\sigma(I)$, we conclude that the assumption (H2) is satisfied with $c = F(\hat{X})$.

Claim 5.6 There exist $\tilde{\delta} > 0$ such that

$$\|\text{grad } F_j(X)\| > \tilde{\delta}, \quad \forall X \in S_F(F(Q)), \quad j = 1, 2, 3, \quad (44)$$

where Q is defined in (39).

Proof Let $X \in M$. Using (29) along with the definition of the Riemannian metric, we have:

$$\begin{aligned} \|\text{grad } F_1(X)\| &= \sqrt{n}, \\ \|\text{grad } F_2(X)\| &= \sqrt{\text{tr}(I + 4e^{-2\text{tr } X} \text{tr } X + 4e^{-4\text{tr } X} \text{tr } X^2)}, \\ \|\text{grad } F_3(X)\| &= \sqrt{\text{tr } X^{-2}}. \end{aligned}$$

Now, if $X \in S_F(F(Q))$, then from the second inequality in (40) we conclude that:

$$\|\text{grad } F_3(X)\| > \sqrt{n}/\kappa.$$

It is easily seen that $\|\text{grad } F_2(X)\| > \sqrt{n}$. Thus, the result follows by taking $\tilde{\delta} = \sqrt{n}/\kappa$. ■

For checking that F satisfies the assumption (H3) we will use the following notation:

$$B_r^E(I) := \{X \in S_{++}^n : \|X - I\|_F < r\}, \quad (45)$$

for the Euclidean ball with centre in I and radius r . Since $I \in \text{int}S_F(F(\hat{X}))$, where \hat{X} is as defined in Claim 5.5, take $r > 0$ such that

$$B_r^E(I) \subset S_F(F(\hat{X})).$$

Using Claim 5.5 we have $S_F(F(\hat{X})) \subset S_F(F(Q))$, with Q as in (39). Then in order to prove that F satisfies the assumption (H3) it is sufficient to prove the following claim:

Claim 5.7 There exists $\delta > 0$ such that

$$\|y(X)\| > \delta, \quad X \in S_F(F(Q)) \setminus B_r^E(I), \quad y(X) \in \partial^\circ F(X). \quad (46)$$

Proof Let $B_r^E(I)$ be the Euclidean ball as defined in (45). Take $X \in S_F(F(Q)) \setminus B_r^E(I)$ and consider the active index set

$$I(X) := \{i : F(X) = F_i(X), i = 1, 2, 3\},$$

of F at X . Hence Claim 5.1 implies that

$$I(X) = \begin{cases} \{3\}, & X \in U_1, \\ \{1\}, \{3\} \text{ or } \{1, 3\}, & X \in U_2, \\ \{2\}, \{3\} \text{ or } \{2, 3\}, & X \in U_3. \end{cases} \quad (47)$$

So, we conclude from Claim 5.6 that in order to prove the claim, it suffices to consider the following cases:

- (a) $X \in D_1 = \{X \in U_2 : I(X) = \{1, 3\}\}$;
- (b) $X \in D_2 = \{X \in U_3 : I(X) = \{2, 3\}\}$.

Let us suppose, initially, that (a) holds, i.e. $X \in D_1$. Take $y(X) \in \partial^\circ F(X)$. Lemma A.3 in Appendix guarantees that there exists $\alpha \in [0, 1]$ such that:

$$y(X) = F_1(X) + (1 - \alpha) \text{grad } F_3(X).$$

Now, using (29), the definition of the Riemannian metric and the latter expression, we have:

$$\|y(X)\|^2 = \text{tr}(\alpha I - (1 - \alpha)X^{-1})^2,$$

which after simple algebraic manipulations becomes

$$\|y(X)\|^2 = \sum_{j=1}^n (\alpha - (1 - \alpha)\lambda_j^{-1}(X))^2, \quad (48)$$

where $\lambda_j(X)$ denotes the j th eigenvalue of the matrix X . From (48) it is easy to see that

$$\|y(X)\|^2 \geq (\alpha - (1 - \alpha)\lambda_{\min}^{-1}(X))^2 + (\alpha - (1 - \alpha)\lambda_{\max}^{-1}(X))^2.$$

Minimizing the second degree polynomial on the right-hand side in the above inequality with respect to α , we obtain:

$$\|y(X)\|^2 \geq \frac{(\lambda_{\min}^{-1}(X) - \lambda_{\max}^{-1}(X))^2}{(1 + \lambda_{\min}^{-1}(X))^2 + (1 + \lambda_{\max}^{-1}(X))^2}. \quad (49)$$

Since $X \in D_1 \subset U_2$, combining Claim 5.1 with Claim 5.2, we have $\lambda_{\min}(X) \leq 1$. Furthermore, in this case $\det X > 1$ which, combining with $\lambda_{\min}(X) \leq 1$, implies that

$$\lambda_{\max}(X) > 1.$$

On the other hand, as $\det X > 1$ Equation (34) yields $\operatorname{tr} X > n$. Since $X \notin B_r^E(I)$ and $\operatorname{tr} X > n$, we conclude from the above inequality and after some algebraic manipulations that

$$r^2 \leq \operatorname{tr}(X - I)^2 = \operatorname{tr} X^2 - 2\operatorname{tr} X + n \leq n\lambda_{\max}^2(X) - 2n + n\lambda_{\max}^2(X),$$

which implies that

$$\lambda_{\max}(X) > \sqrt{1 + r^2/2n}.$$

Combining first the inequality in (40) with the inequalities (49) and the latter inequality, we obtain:

$$\|y(X)\| > \delta_1 = \frac{1 - 1/\sqrt{1 + r/2n}}{\sqrt{(1 + \kappa)^2 + (1 + 1/\sqrt{1 + r/2n})^2}} > 0, \quad X \in D_1, \quad y(X) \in \partial^\circ F(X). \quad (50)$$

Now, let us suppose that the item (b) holds, i.e. $X \in D_1$. Take $y(X) \in \partial^\circ F(X)$. Due to Lemma A.3 there exists $\alpha \in [0, 1]$ such that

$$y(X) = F_1(X) + (1 - \alpha) \operatorname{grad} F_3(X).$$

Thus, (29), definition of the Riemannian metric and the last expression imply that

$$\|y(X)\|^2 = \operatorname{tr}(\alpha(-X - 2e^{-2\operatorname{tr} X} X^2) - (1 - \alpha)I)^2.$$

Some algebraic manipulations in the last equality and taking into account that $\lambda_j(X) > 0$, for all $j = 1, \dots, n$ yields:

$$\|y(X)\|^2 > \alpha^2 n + 2\alpha(1 - \alpha)\operatorname{tr} X^{-1} + (1 - \alpha)\operatorname{tr} X^{-2}. \quad (51)$$

Due to $X \in D_2 \subset U_3$ we have $\det X^{-1} > 0$. Hence, using (34), we obtain:

$$\operatorname{tr} X^{-1} > n, \quad \operatorname{tr} X^{-2} > n.$$

Since $\alpha \in [0, 1]$, substituting the above two inequalities into (51) we obtain

$$\|y(X)\|^2 > n, \quad X \in D_2, \quad y(X) \in \partial^\circ F(X). \quad (52)$$

Therefore, taking $\delta = \min\{\tilde{\delta}, \delta_1, n\} > 0$, where $\tilde{\delta}$ is given in the last claim, the claim follows from (47), (50), (52) and Claim 5.6. \blacksquare

Thus $S_F(F(Q)) \setminus S_F(F(\hat{X})) \subset S_F(F(Q)) \setminus B_r^E(I)$, the assumption (H3) is satisfied with $\delta = \min\{\tilde{\delta}, \delta_1, \delta_2\} > 0$.

Summarizing, all assumptions of Theorem 4.1 are satisfied by taking $\Omega = \{X \in \mathbb{S}_{++}^n : \operatorname{tr} X^{-1} < \kappa n\}$, $q = \operatorname{diag}(1/4, \dots, 1/4)$, $c = F(\hat{X})$, where \hat{X} is as defined in Claim 5.5, and $\delta > 0$ is defined above. Letting $X^0 \in \mathbb{S}_{++}^n$ and $\bar{\lambda} > 0$ such that $X^0 \in S_f(f(q))$ and $\max_{i \in I} L_i < \lambda_k \leq \bar{\lambda}$, the proximal point method may be applied for solving the above nonconvex problem (32).

Remark 5.2 In agreement with Claims 5.1 and 5.2, the function

$$F(X) = \max\{\ln \det X, -4 \ln \det X + e^{-2\operatorname{tr} X} - e^{-2n}, \operatorname{tr} X^{-1} - n\},$$

in the above example, is nonconvex (in Euclidean sense) when restricted to any open convex neighbourhood containing its minimizer $X^* = I$. Therefore, the classical local proximal point method cannot be applied to minimize this function. From the second relationship in (30) together with Claim 5.3 we conclude that the function F is nonconvex in the Riemannian sense as well. Therefore, the Riemannian proximal point method also cannot be applied to minimize this function either.

6. Final remarks

We have extended the range of application of the proximal point method to solve nonconvex optimization problems on Hadamard manifolds, namely, in the case of the objective function being given by the maximum of a certain class of continuously differentiable functions. We have certified, through examples, that the class of minimization problems for which the local proximal point method can be applied is different from the class of minimization problem for which the classical local proximal point and the Riemannian proximal point method are applied. An interesting subject now is to extend the proximal point method to minimization problems where objective functions are lower- C^2 type.

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Appendix

In this section we will use notation as in Example 5.2.

LEMMA A.1 *If $X \in M$ and $V \in T_X M$, then*

$$\lim_{t \rightarrow 0} \lim_{Y \rightarrow X} \frac{d(\exp_Y t(D \exp_X)_{\exp_X^{-1} Y} V, Y + tV)}{t} = 0.$$

Proof Preliminarily, note that

$$d(\exp_Y t(D \exp_X)_{\exp_X^{-1} Y} V, Y + tV) = d(\exp_Y t(D \exp_X)_{\exp_X^{-1} Y} V, \exp_Y(\exp_Y^{-1}(Y + tV))).$$

Since the exponential map $\exp: TM \rightarrow M$ is smooth, it is in particular locally Lipschitz. Let $U_X \subset M$ be a neighbourhood of X such that $TU_X \approx U_X \times T_X M$ and the map \exp is Lipschitz in TU_X with constant K . Take $\delta = \delta(X) > 0$ such that, for all $Y \in B_\delta(X)$ and $t \in (0, \delta)$,

$$d(\exp_Y t(D \exp_X)_{\exp_X^{-1} Y} V, \exp_Y(\exp_Y^{-1}(Y + tV))) \leq K \|\mathcal{H}(t, Y)\|, \quad (53)$$

where $\mathcal{H}(t, Y) = t(D \exp_X)_{\exp_X^{-1} Y} V - \exp_Y^{-1}(Y + tV)$. On the other hand, by the definition of the metric (26)

$$\|\mathcal{H}(t, Y)\|^2 = \text{tr} \left(\left(t(D \exp_X)_{\exp_X^{-1} Y} V - \exp_Y^{-1}(Y + tV) \right) Y^{-1} \right)^2.$$

The definition of \exp_Y^{-1} in (27) and above equality yield:

$$\|\mathcal{H}(t, Y)\|^2 \leq \left[\left(t(D \exp_X)_{\exp_X^{-1} Y} V - Y^{1/2} \ln(I + tY^{-1/2} V Y^{-1/2}) Y^{1/2} \right) Y^{-1} \right]^2,$$

which, after simple algebraic manipulation, implies:

$$\left\| \frac{\mathcal{H}(t, Y)}{t} \right\|^2 \leq \left[\left((D \exp_X)_{\exp_X^{-1} Y} V - Y^{1/2} \ln(I + tY^{-1/2} V Y^{-1/2})^{1/t} Y^{1/2} \right) Y^{-1} \right]^2. \quad (54)$$

It can be easily seen that:

$$\lim_{Y \rightarrow X} (D \exp_X)_{\exp_X^{-1} Y} V = V, \quad \lim_{t \rightarrow 0} \lim_{Y \rightarrow X} \ln(I + tY^{-1/2} V Y^{-1/2})^{1/t} = X^{-1/2} V X^{-1/2}.$$

Hence, combining inequality (54) with the latter two equalities, we conclude that:

$$\lim_{t \rightarrow 0} \lim_{Y \rightarrow X} \left\| \frac{\mathcal{H}(t, Y)}{t} \right\| = 0.$$

The last equality along with (53) implies the desired result. \blacksquare

LEMMA A.2 *Let $\Omega \subset \mathbb{S}_{++}^n$ be an open convex set. If F is a locally Lipschitz function on Ω , then*

$$F^\circ(X, V) = F_x^\circ(X, V), \quad X \in \Omega, \quad V \in T_X M.$$

Proof Take $X \in \Omega$ and $V \in T_X M$. Since F is locally Lipschitz on Ω and TM is locally a product, there exists $\delta = \delta(X) > 0$ such that $TB_\delta(X) \approx B_\delta(X) \times \mathbb{R}^n$ and

$$\begin{aligned} & \left| \frac{F(\mathcal{G}(t, Y)) - F(Y + tV)}{t} \right| \\ & \leq L_X \frac{d(\mathcal{G}(t, Y), Y + tV)}{t}, \quad \mathcal{G}(t, Y) = \exp_Y t(D \exp_X)_{\exp_X^{-1} Y} V, \end{aligned}$$

for all $Y \in B_\delta(X)$, $t \in (0, \delta)$. Note that the above inequality is equivalent to

$$\left| \frac{F(\mathcal{G}(t, Y)) - F(Y)}{t} - \frac{F(Y + tV) - F(Y)}{t} \right| \leq L_X \frac{d(\mathcal{G}(t, Y), Y + tV)}{t}.$$

On the other hand, Lemma A.1 implies that

$$\lim_{t \rightarrow 0, q \rightarrow p} \frac{d(\mathcal{G}(t, Y), Y + tV)}{t} = 0,$$

which combined with the above inequality and definitions of the generalized derivatives yields the lemma. ■

LEMMA A.3 *Let $\Omega \subset \mathbb{S}_{++}^n$ be an open convex set and $I = \{1, \dots, m\}$. Let $F_i: M \rightarrow \mathbb{R}$ be a continuously differentiable function on Ω for every $i \in I$ and $F: M \rightarrow \mathbb{R}$ defined by*

$$F(X) := \max_{i \in I} F_i(X).$$

Then F is locally Lipschitz on Ω and the following holds:

$$\partial^\circ F(X) = \text{conv} \{ \text{grad } F_i(X) : i \in I(X) \}, \quad X \in \Omega,$$

where $I(X) := \{i: F_i(X) = F(X), i = 1, \dots, m\}$.

Proof Take $X \in \Omega$. From Proposition 3.4 it is sufficient to prove that

$$\partial^\circ F(X) \{ \text{grad } F_i(X) : i \in I(X) \}.$$

Let $W \in \partial^\circ F(X)$. Then, the definition of the set $\partial^\circ F(X)$ and (26) provide

$$F^\circ(X, V) \geq \langle W, V \rangle = \text{tr}(V \Psi''(X) W), \quad V \in \mathbb{S}^n,$$

where $\Psi(X) = -\ln \det X$. Combining Lemma A.2 with the definition of the Euclidean generalized subdifferential of F at X , $\partial_E^\circ F(X)$, we conclude that

$$\Psi''(X) W \in \partial_E^\circ F(X).$$

On the other hand,

$$\partial_E^\circ F(X) \{ \nabla F_i(X) : i \in I(X) \},$$

where $I(X) := \{i \in I: F_i(X) = F(X)\}$, see [11, Proposition 2.3.12]. Then, there exists a constant $\alpha_i \geq 0$ for $i \in I(X)$ with $\sum_{i \in I(X)} \alpha_i = 1$ such that

$$\Psi''(X) W = \sum_{i \in I(X)} \alpha_i \nabla F_i(X).$$

So, since $\Psi''(X)$ is invertible and $\text{grad } F_i(X) = \Psi''(X)^{-1} \nabla F_i(X)$, the above equality yields

$$W = \sum_{i \in I(X)} \alpha_i \text{grad } F_i(X).$$

Hence $W \in \text{conv} \{ \text{grad } F_i(X) : i \in I(X) \}$ and the result is proved. ■