GENERALIZED VECTOR EQUILIBRIUM PROBLEMS AND ALGORITHMS FOR VARIATIONAL INEQUALITY IN HADAMARD MANIFOLDS

Doctoral Thesis by

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Abstract

In this thesis, we study variational inequalities and generalized vector equilibrium problems.

In Chapter 1, several results and basic definitions of Riemannian geometry are listed; we present the concept of the monotone vector field in Hadamard manifolds and many of their properties, besides, we introduce the concept of enlargement of a monotone vector field, and we display its properties in a Riemannian context.

In Chapter 2, an inexact proximal point method for variational inequalities in Hadamard manifolds is introduced, and its convergence properties are studied; see [7]. To present our method, we generalize the concept of enlargement of monotone operators, from a linear setting to the Riemannian context. As an application, an inexact proximal point method for constrained optimization problems is obtained.

In Chapter 3, we present an extragradient algorithm for variational inequality associated with the point-to-set vector field in Hadamard manifolds and study its convergence properties; see [8]. In order to present our method, the concept of enlargement of maximal monotone vector fields is used and its lower-semicontinuity is established to obtain the convergence of the method in this new context.

In Chapter 4, we present a sufficient condition for the existence of a solution to the generalized vector equilibrium problem on Hadamard manifolds using a version of the Knaster-Kuratowski-Mazurkiewicz Lemma; see [6]. In particular, the existence of solutions to optimization, vector optimization, Nash equilibria, complementarity, and variational inequality is a special case of the existence result for the generalized vector equilibrium problem.

Keywords: Enlargement of vector fields; inexact proximal; constrained optimization; extragradient algorithm; lower-semicontinuity; vector equilibrium problem; vector optimization; Hadamard manifold.

Resumo

Nesta tese, estudamos desigualdades variacionais e o problema de equilíbrio vetorial generalizado.

No Capítulo 1, vários resultados e definições elementares sobre geometria Riemanniana são enunciados; apresentamos o conceito de campo vetorial monótono e muitas de suas propriedades, além de introduzir o conceito de alargamento de um campo vetorial monótono e exibir suas propriedades em um contexto Riemanniano.

No Capítulo 2, um método de ponto proximal inexato para desigualdades variacionais em variedades de Hadamard é introduzido e suas propriedades de convergência são estudados; veja [7]. Para apresentar o nosso método, generalizamos o conceito de alargamento de operadores monótonos, do contexto linear ao contexto de Riemanniano. Como aplicação, é obtido um método de ponto proximal inexato para problemas de otimização com restrições.

No Capítulo 3, apresentamos um algoritmo extragradiente para desigualdades variacionais associado a um campo vetorial ponto-conjunto em variedades de Hadamard e estudamos suas propriedades de convergência; veja [8]. A fim de apresentar nosso método, o conceito de alargamento de campos vetoriais monótonos é utilizado e sua semicontinuidade inferior é estabelecida, a fim de obter a convergência do método neste novo contexto.

No Capítulo 4, apresentamos uma condição suficiente para a existência de soluções para o problema de equilíbrio vetorial generalizado em variedades de Hadamard usando uma versão do Lema Knaster-Kuratowski-Mazurkiewicz; veja [6]. Em particular, a existência de soluções para problemas de otimização, otimização vetorial, equilíbrio de Nash, complementaridade e desigualdades variacionais são casos especiais do resultado de existência do problema de equilíbrio vetorial generalizado.

Palavras-chave : Alargamento de campos vetoriais; proximal inexato; otimização com restrições; algoritmo extragradiente; semicontinuidade inferior; problema de equilíbrio vetorial; otimização vetorial; variedade de Hadamard.

Contents

Introduction

_Indrodction⟩?

In recent years, there has been an increasing number of studies proposing extensions of concepts, techniques, and methods of mathematical programming pertaining to the linear setting to the Riemannian context; papers published on this topic involving proximal methods include [2,5,10,49,54,59]. It is well known that convexity and monotonicity play an important role in the analysis and development of methods of mathematical programming. One of the reasons for the extension of this concepts from the linear to the Riemannian setting is the possibility of transforming non-convex or non-monotone problems in the Euclidean context into Riemannian convex or monotone problems, by introducing a suitable metric, which enables modified numerical methods to find solutions for these problems; see [10,11,19,22,50]. These extensions, which in general are nontrivial, are either of a purely theoretical nature or they aim at obtaining numerical algorithms. Indeed, many mathematical programming problems are naturally posed on Riemannian manifolds having specific underlying geometric and algebraic structure that could also be exploited to reduce the cost of obtaining the solutions; see, e.g., [1,25,38,48,52].

In this study, we consider the problem of finding a solution of variational inequalities defined on Riemannian manifolds. Variational inequalities on Riemannian manifolds were first introduced and studied by Németh in [47], for univalued vector fields on Hadamard manifolds, and for multivalued vector fields on general Riemannian manifolds by Li and Yao in [42]; for recent works addressing this subject see [27,43,55,56]. It is worth noting that constrained optimization problems and the problem of finding the zero of a multivalued vector field, which were studied in [2,10,23,29,40,59], are particular instances of the variational inequality. We also consider the generalized vector equilibrium problem in Riemannian context and obtain the existence of solution for this problem.

In [17] an inexact proximal point algorithm to solve variational inequalities point-to-set in Euclidean spaces with Bregman distances was introduced. In order to ensure the well-definition and quasi-Fejér convergence of the algorithm to solve $VIP(T;C)$, the operator T was supposed to be maximal monotone. The monotonicity of an operator is a classical hypothesis; for example when $T = \partial f$ is the subdifferential of a function f, the $VIP(T;C)$ is equivalent to the problem of minimizing f restrict to C, and the monotonicity of the operator T is equal the convexity of the objective function f. In turn, the maximality of the operator T is also a standard assumption in the variational inequalities approach with the point-to-set operator T; it makes the role of continuity in the case where T is point-to-point. To ensure the convergence of method, another assumption of a more technical nature has been introduced, namely, the paramonotonicity of the operator.

In [40], a proximal point algorithm was introduced to find singularity of vector fields in Hadamard manifolds, extending the classical method in Euclidean spaces. The major difficulty in this mentioned work, was to establish the well-definition of the method in the Riemannian context. This goal was achieved thanks to new properties (in a Riemannian context) established for maximal monotone vector fields, plus the assumption that the vector field is everywhere defined, something that is not required in linear spaces.

Motivated by these two studies, we proposed in [7] a proximal point method to solve variational inequalities that generalizes these two papers above. Regarding [17], the generalization is of linear context to the Riemannian, plus we suppress the paramonotonicity assumption. As to the [40], we kept the same assumptions about the vector field, but obtained an inexact version of the method by introducing, in the context Riemannian, the enlargement of a monotone vector field X. Represented here by X^{ϵ}, the enlargement of monotone operator X resembles the ϵ-subdifferential of a convex function, both enjoy the property to contain, at each point, the image of the operator originally associated. In [7], we have extended many properties of the enlargement of a monotone operator stated in [17] to the Riemannian context.

One of the most important properties of T^{ϵ} is the lower semicontinuity. We remark that even when T is maximal monotone, it can not enjoy the lower semicontinuity property. Iusem and Pérez leave explicit the importance of lower semicontinuity of the T^{ϵ} operator in the convergence analysis of extragradient algorithm presented in [36]. It is this property that provides the convergence of the method, without adding too restrictive assumptions over T operator, such as strong monotonicity or Lipschitz-continuity. In [8] we extended the lower semicontinuity of X^{ϵ} to Riemannian context and we present an extragradient method for solving variational inequalities in Hadamard manifolds that generalizes the method presented in [36]. Furthermore, in the case where X is point-to-point, our method retrieves as a special case (under the assumptions of continuity and monotonicity of the vector field X and still assuming $\epsilon_k = 0$, for all k), the algorithms presented in [53] and [30].

The generalized vector equilibrium problem (GVEP) in Riemannian context was also the subject of our study. The GVEP has been widely studied and continues to be an active topic of research. One of the primary reasons for this is that multiple problems can be formulated as GVEP, such as optimization, vector optimization, Nash equilibria, complementarity, fixed point, and variational inequality problems. Extensive developments of these problems can be found in Fu [31], Fu and Wan [32], Konnov and Yao [39], Ansari et al. [3], Farajzadeh et al. [28], and the references therein. An important question concerns the conditions under which there exists a solution to the GVEP. In a linear setting, multiple authors have provided results that answer this question, such as Ansari and Yao [4], Fu [31], Fu and Wan [32], Konnov and Yao [39], Ansari et al. [3], Farajzadeh et al. [28] and the authors referenced in their work. Colao et al. [19] and Zhou and Huang [62] were the first to examine the existence of solutions for equilibrium problems in the Riemannian context by generalizing the Knaster-Kuratowski-Mazurkiewicz (KKM) Lemma to a Hadamard manifold. Applying the KKM Lemma in a Riemannian setting allowed Zhou and Huang [44] to confirm solution existence for vector optimization problems and vector variational inequalities in this context. Similarly, Li and Huang [61] presented results concerning solution existence for a special class of GVEP. In [6], we apply the KKM Lemma in a Riemannian setting to prove existence of solution for the GVEP. It should be noted that our results include the results presented in [19, 44] and are not included in [61].

Chapter 1

Basic Results in Riemannian Manifolds

⟨chapter1⟩

In this chapter, we introduce some fundamental properties and notations about Riemannian geometry. These basics facts can be found in any introductory book on Riemannian geometry, such as [24, 51].

We denote by T_pM the n-dimentional *tangent space* of M at p, by $TM = \cup_{p\in M} T_pM$ *tangent bundle* of M and by $\mathcal{X}(M)$ the space of smooth vector fields on M. The Riemannian metric is denoted by $\langle\ ,\ \rangle$ and the corresponding norm by $\|\ \|$. Denote the length of piecewise smooth curves $\gamma : [a, b] \to M$ joining p to q, i.e., $\gamma(a) = p$ and $\gamma(b) = q$, by

$$l(\gamma) = \int_a^b \|\gamma'(t)\| dt,$$

and the Riemannian distance by $d(p, q)$, which induces the original topology on M, namely, (M, d) is a complete metric space and the bounded and closed subsets are compact. For $A \subset M$, the notation $\mathrm{int}(A)$ implies the interior of A, and if A is a nonempty set, the distance from $p \in M$ to A is given by $d(p, A) := \inf\{d(p, q)\ :\ q \in A\}$. The metric induces a map $f \mapsto \mathrm{grad}\, f \in \mathcal{X}(M)$, which associates to each smooth function f over M its gradient via the rule $\langle \mathrm{grad}\, f, X\rangle = df(X),\ X \in \mathcal{X}(M)$. Let ∇ be the Levi-Civita connection associated to $(M, \langle\ ,\ \rangle)$. A vector field V along γ is said to be *parallel* iff $\nabla_{\gamma'} V = 0$ and the parallel transport along γ from p to q is denoted by $P_{pq} : T_pM \to T_qM$. If γ' itself is parallel, we say that γ is *geodesic*. Given that the geodesic equation $\nabla_{\ \gamma'}\gamma' = 0$ is a second order nonlinear ordinary differential equation, the geodesic $\gamma = \gamma_v(., p)$ is determined by its position p and velocity v at p. It is simple to check if $\|\gamma'\|$ is constant. We say that γ is *normalized* iff $\|\gamma'\| = 1$. The restriction of a geodesic to a closed bounded interval is called a *geodesic segment*. In this thesis, all manifolds M are assumed to be Hadamard finite dimensional, then the length of the geodesic segment γ joining p to q equals $d(p, q)$ and the *exponential map*

$\exp_p : T_pM \to M$ defined by $\exp_p v = \gamma_v(1, p)$ is a diffeomorphism and, consequently, M is diffeomorphic to the Euclidean space $\mathbb{R}^n$, $n = \dim M$. Let $q \in M$ and $\exp_q^{-1} : M \to T_pM$ be the inverse of the exponential map. Note that $d(q\,,\, p) = ||\exp_p^{-1} q||$, the map $d_q^2 : M \to \mathbb{R}$ defined by $d_q^2(p) = d^2(q, p)$ is C^∞ and

$$\operatorname{grad} d_q^2(p) := -2 \exp_p^{-1} q. \tag{1.1}$$ eq:gd2

Furthermore, we know that

$$d^2(p_1, p_3) + d^2(p_3, p_2) - 2\langle \exp_{p_3}^{-1} p_1, \exp_{p_3}^{-1} p_2 \rangle \le d^2(p_1, p_2), \qquad p_1, p_2, p_3 \in M. \tag{1.2}$$ eq:coslaw

$$\langle \exp_{p_2}^{-1} p_1,\, \exp_{p_2}^{-1} p_3 \rangle + \langle \exp_{p_3}^{-1} p_1,\, \exp_{p_3}^{-1} p_2 \rangle \ge d^2(p_2, p_3), \qquad p_1, p_2, p_3 \in M. \tag{1.3}$$ eq:coslaw2

A set $\Omega \subseteq M$ is said to be *convex* iff any geodesic segment with end points in Ω is contained in Ω. Given an arbitrary set $\mathcal{B} \subset M$, the minimal convex set containing $\mathcal{B}$ is called the *convex hull* of $\mathcal{B}$ and is denoted by $\operatorname{conv}(\mathcal{B})$; see [19].

Let $\Omega \subset M$ be a closed, convex set and $p \in M$. Thus the projection $\mathrm{P}_\Omega(p)$ of p onto Ω satisfies

$$\left\langle \exp_{\mathrm{P}_\Omega(p)}^{-1} q,\, \exp_{\mathrm{P}_\Omega(p)}^{-1} p \right\rangle \le 0, \qquad q \in M, \tag{1.4}$$ eq:proj

see [29, Corollary 3.1].

The projection onto a nonempty, closed and convex subset $\Omega \subset M$ is nonexpansive, i.e. there holds

$$d(\mathrm{P}_\Omega(p), \mathrm{P}_\Omega(q)) \le d(p, q) \qquad p, q \in \Omega, \tag{1.5}$$ eq:projnon-ex

see [41, Corollary 1].

⟨eq:ContExp⟩ **Lemma 1.0.1** *Let $\bar p, \bar q \in M$ and $\{p^n\}, \{q^n\} \subset M$ be such that $p^n \to \bar p$ and $q^n \to \bar q$. Then the following assertions hold.*

i) For any $q \in M$, we have

$$\exp_{p^n}^{-1} q \longrightarrow \exp_{\bar p}^{-1} q \quad \textit{and} \quad \exp_q^{-1} p^n \longrightarrow \exp_q^{-1} \bar p.$$

ii) If $v^n \in T_{p^n}M$ and $v^n \to \bar v$, then $\bar v \in T_{\bar p}M$.

iii) For any $u \in T_{\bar p}M$, the function $F : M \to TM$ defined by $F(x) = \mathrm{P}_{\bar{\mathrm{p}}\mathrm{p}}\mathrm{u}$ each $p \in M$ is continuous on M.

iv) $\exp_{p^n}^{-1} q^n \longrightarrow \exp_{\bar p}^{-1} \bar q$.

Proof. For items (i), (ii) and (iii) see [40, Lemma 2.4]. For the item (iv), using triangular inequality we obtain

$$\| \exp_{p^n}^{-1} q^n - \mathrm{P}_{\bar{p}p^n} \exp_{\bar{p}}^{-1} \bar{q} \| \leq \| \exp_{p^n}^{-1} q^n - \exp_{p^n}^{-1} \bar{q} \| + \| \exp_{p^n}^{-1} \bar{q} - \mathrm{P}_{\bar{p}p^n} \exp_{\bar{p}}^{-1} \bar{q} \| .$$

Since M have nonpositive curvature we have $\| \exp_{p^n}^{-1} q^n - \exp_{p^n}^{-1} \bar{q} \| \leq d(q^n, q)$. It follows then that

$$\| \exp_{p^n}^{-1} q^n - \mathrm{P}_{\bar{p}p^n} \exp_{\bar{p}}^{-1} \bar{q} \| \leq d(q^n, \bar{q}) + \| \exp_{p^n}^{-1} \bar{q} - \mathrm{P}_{\bar{p}p^n} \exp_{\bar{p}}^{-1} \bar{q} \| .$$

Taking limit with $n \to \infty$ in the last inequality and combining items (i) and (iii) we can conclude that $\exp_{p^n}^{-1} q^n \longrightarrow \exp_{\bar{p}}^{-1} \bar{q}$. ■

Let $\Omega \subset M$ be a convex set and $p \in \Omega$. We define the *normal cone* to Ω at p (see [40]) by

$$N_\Omega(p) := \left\{ w \in T_pM \ : \ \left\langle w, \exp_p^{-1} q \right\rangle \leq 0, q \in \Omega \right\}. \tag{1.6}$$ eq:nc

The *domain* of $f : M \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\mathrm{dom} f := \{ p \in M \ : \ f(p) < \infty \} .$$

The function f is said to be proper iff $\mathrm{dom} f \neq \varnothing$ and it is *convex* on a convex set $\Omega \subset \mathrm{dom} f$ iff for any geodesic segment $\gamma : [a, b] \to \Omega$, the composition $f \circ \gamma : [a, b] \to \mathbb{R}$ is convex. It is well known that d_q^2 is convex. Consider $p \in \mathrm{dom} f$. A vector $s \in T_pM$ is said to be a *subgradient* of f at p iff

$$f(q) \geq f(p) + \left\langle s, \exp_p^{-1} q \right\rangle , \qquad q \in M.$$

The set $\partial f(p)$ of all subgradients of f at p is called the *subdifferential* of f at p. The function f is *lower semicontinuous* at $\bar{p} \in \mathrm{dom} f$ iff for each sequence $\{p^k\}$ converging to $\bar{p}$, we have

$$\liminf_{k \to \infty} f(p^k) \geq f(\bar{p}).$$

Given a multivalued vector field $X : M \rightrightarrows TM$, the domain of X is the set defined by

$$\mathrm{dom} X := \{ p \in M \ : \ X(p) \neq \varnothing \} .$$

Let $X : M \rightrightarrows TM$ be a vector field and $\Omega \subset M$. We define the following quantity

$$m_X(\Omega) := \sup_{q \in \Omega} \{ \|u\| \ : \ u \in X(q) \} .$$

We say that X is *locally bounded* iff for all $p \in \mathrm{int}(\mathrm{dom}\, X)$ there exist an open set $U \subset M$ such that $p \in U$ and there holds $m_X(U) < +\infty$, and *bounded on bounded sets* iff for all bounded set $V \subset M$ such that its closure $\overline{V} \subset \mathrm{int}(\mathrm{dom}\, X)$ it holds that $m_X(V) < +\infty$;

see an equivalent definition in [40]. The multivalued vector field X is said to be *upper semicontinuous* at $p \in \mathrm{dom}X$ iff, for any open set $V \subset T_pM$ such that $X(p) \in V$, there exists an open set $U \subset M$ with $p \in U$ such that $P_{qp}X(q) \subset V$, for any $q \in U$. For two multivalued vector fields X, Y on M, the notation $X \subset Y$ implies that $X(p) \subset Y(p)$, for all $p \in M$.

For any set $\mathcal{A}$, we let $2^{\mathcal{A}}$ represent the set of all subsets of $\mathcal{A}$. Let $\Omega \subseteq M$ be a nonempty set and $\mathbb{Y}$ a topological vector space. Given a set valued mapping $T : \Omega \rightrightarrows \mathbb{Y}$, the domain and range are the sets respectively defined by the following:

$$\mathrm{dom}T := \{x \in \Omega \ : \ T(x) \neq \varnothing\}, \qquad \mathrm{rge}\, T := \{y \in \mathbb{Y} \ : \ y \in T(x) \text{ for some } x \in \Omega\}. \quad (1.7)$$ eq:dr

Moreover, the *inverse* of T is the set-valued mapping $T^{-1} : \mathbb{Y} \rightrightarrows \Omega$ defined by

$$T^{-1}(y) := \{x \in \Omega \ : \ y \in T(x)\}. \quad (1.8)$$ eq:dr

The following result is a version of the KKM lemma in Riemannian context due to [19], which is an extension of KKM theorem that can be found, for example, in [57].

⟨le:colao⟩ **Lemma 1.0.2** *Let $\Omega \subseteq M$ be a nonempty, closed and convex set, and $G : \Omega \rightrightarrows \Omega$ a set-valued mapping such that, for each $y \in \Omega, G(y)$ is closed. Suppose that there exists $y_0 \in \Omega$ such that $G(y_0)$ is compact and, for all $y_1, ..., y_m \in \Omega$, we have* $\mathrm{conv}(\{y_1, ..., y_m\}) \subset \bigcup_{i=1}^m G(y_i)$. *Then $\bigcap_{y\in\Omega} G(y) \neq \varnothing$.*

Proof. See [19, Lemma 3.1]. ■

A sequence $\{p^k\} \subset (M, d)$ is said to *Fejér convergent* to a nonempty set $W \subset M$ iff, for every $q \in W$, we have $d^2(q, p^{k+1}) \leq d^2(q, p^k)$, for $k = 0, 1, \ldots$

⟨fejer⟩ **Proposition 1.0.3** *Let $\{p^k\}$ be a sequence in (M, d). If $\{p^k\}$ is Fejér convergent to nonempty set $W \subset M$, then $\{p^k\}$ is bounded. If furthermore, an accumulation point p of $\{p^k\}$ belongs to W, then $\lim_{k\to\infty} p^k = p$.*

A sequence $\{p^k\} \subset (M, d)$ is said to be *quasi-Fejér convergent* to a nonempty set $W \subset M$ iff, for every $q \in W$ there exists a summable sequence $\{\epsilon_k\} \subset \mathbb{R}_+$, such that $d^2(q, p^{k+1}) \leq d^2(q, p^k) + \epsilon_k$, for $k = 0, 1, \ldots$; see Burachik et al. [14].

We need of following result, whose proof is analogous to the proof of [14, Theorem 1], by replacing the Euclidean distance with the Riemannian distance.

⟨quasi-fejer⟩? **Proposition 1.0.4** *Let $\{p^k\}$ be a sequence in (M, d). If $\{p^k\}$ is quasi-Fejér convergent to the nonempty set $W \subset M$, then $\{p^k\}$ is bounded. If furthermore, an accumulation point p of $\{p^k\}$ belongs to W, then $\lim_{k\to\infty} p^k = p$.*

⟨lemmaseq.⟩

Lemma 1.0.5 *Let $\{\rho_k\}$ be a sequence of positive real numbers and $\theta_0 > 0$. Define the sequence $\{\theta_k\}$ by $\theta_{k+1} = \min\{\theta_k, \rho_k\}$. The limit $\bar{\theta}$ of $\{\theta_k\}$ is equal to 0 if and only if 0 is a cluster point of $\{\rho_k\}$.*

Proof. See [36, Lemma 4.9]. ■

1.1 Monotone vector fields

We begin by recalling the notions of monotonicity and maximal monotonicity for multivalued vector fields on Hadamard manifolds. A multivalued vector field X is said to be *monotone* iff

$$\left\langle P_{qp}^{-1}u - v, \, \exp_q^{-1} p \right\rangle \geq 0, \qquad p,\, q \in \operatorname{dom} X, \quad u \in X(p), \; v \in X(q), \tag{1.9}$$ eq2.1

and *strongly monotone*, iff there exists $\rho > 0$ such that

$$\left\langle P_{qp}^{-1}u - v, \, \exp_q^{-1} p \right\rangle \geq \rho d^2(p,q), \qquad p,\, q \in \operatorname{dom} X, \quad u \in X(p), \; v \in X(q). \tag{1.10}$$ eq2.2

Moreover, a monotone vector field X is said to be *maximal monotone*, iff for each $p \in \operatorname{dom} X$ and $u \in T_pM$, there holds:

$$\left\langle P_{qp}^{-1}u - v, \, \exp_q^{-1} p \right\rangle \geq 0, \qquad q \in \operatorname{dom} X, \quad v \in X(q) \;\Rightarrow\; u \in X(p).$$

The next definition was introduced in [40].

Definition 1.1.1 *A multivalued vector field X is said to be upper Kuratowski semicontinuous at p if, for any sequences $\{p^k\} \subset \operatorname{dom} X$ and $\{u^k\} \subset TM$, with each $u^k \in X(p^k)$, the relations $\lim_{k\to\infty} p^k = p$ and $\lim_{k\to\infty} u^k = u$ imply that $u \in X(p)$.*

When X is upper Kuratowski semicontinuous for any $p \in \operatorname{dom} X$ we say that X is upper Kuratowski semicontinuous.

op.Kuratowski⟩

Proposition 1.1.2 *Let X a maximal monotone vector field, then X is upper semicontinuous Kuratowski.*

Proof. See Proposition 3.5 of [40]. ■

The concept of monotonicity in the Riemannian context was first introduced in [46], for a single-valued case and in [20] for a multivalued case. Further, the notion of maximal monotonicity of a vector field was introduced in [40]. The next result states that the subdifferential of the convex function is maximal monotone and its proof can be found in [40, Theorem 5.1].

⟨mmsub⟩

Theorem 1.1.3 *Let f be a proper, lower semicontinuous and convex function on M. The subdifferential ∂f is a monotone multivalued vector field. Furthermore, if* $\operatorname{dom} f = M$, *then the subdifferential ∂f of f is a maximal monotone vector field.*

The following lemma is a natural extension to the Riemannian context of the corresponding one in the linear setting.

⟨le:msvf⟩

Lemma 1.1.4 *Let X_1, X_2 be a maximal monotone vector fields such that* $\operatorname{dom} X_1 = \operatorname{dom} X_2 = M$. *Then, $X_1 + X_2$ is a maximal monotone vector field.*

Proof. Let $z \in M$. Define the following operator $T_1, T_2 : T_zM \rightrightarrows T_zM$ by

$$T_1(u) = P_{exp_z u, z} X_1(exp_z u), \qquad T_2(u) = P_{exp_z u, z} X_2(exp_z u),$$

associated to X_1 and X_2, respectively. Since the parallel transport is linear, then there holds

$$(T_1 + T_2)(u) = P_{exp_z u, z}(X_1 + X_2)(exp_z u), \qquad u \in T_zM. \tag{1.11}$$ eq:st1t2

Using that X_1 and X_2 are maximal monotone, then it follows from [40, Theorem 3.7] that T_1 and T_2 are upper semicontinuous, $T_1(u)$ and $T_2(u)$ are closed and convex for each $u \in T_zM$. Thus, we conclude that T_1 and T_2 are maximal monotone, see [18, Theorem 2.5, p. 155]. Since T_1 and T_2 are maximal monotone and $\operatorname{dom}(T_1) = \operatorname{dom}(T_2) = T_zM$, we conclude from [9, Corollary 24.4 (i), p. 353] that $T_1 + T_2$ is maximal monotone. Therefore, combining (1.11) with [40, Theorem 3.7], we conclude that $X_1 + X_2$ is maximal monotone, which concludes the proof. ■

The next two results are extensions to the Hadamard manifolds of its counterpart Euclidean. The proofs of these results are an immediate consequence of the definitions of maximal monotonicity and normal cone and locally bounded vector field.

⟨mon.cone⟩

Lemma 1.1.5 *Let X be a maximal monotone vector field such that* $\operatorname{dom} X = M$. *Then, $X + N_\Omega$ is a maximal monotone vector field.*

Proof. The monotonicity of the $X + N_\Omega$ is immediate from the monotonicity of X and definition of N_Ω. Then, take $p \in M$ and let $u \in T_pM$ be such that

$$-\langle u, \exp_p^{-1} q\rangle - \langle v + w, \exp_q^{-1} p\rangle \geq 0, \qquad q \in M,\ v \in X(q),\ w \in N_\Omega(q). \tag{1.12}$$ {?}

Taking $w = 0$ in last inequality and using the maximality of X we obtain that $u \in X(p)$ and therefore $u + 0 \in (X + N_\Omega)(p)$, which conclude the proof. ■

⟨prop.loc.boun⟩

Proposition 1.1.6 *Suppose X is maximal monotone and* $\operatorname{dom} X = M$. *Then, X is locally bounded on M.*

Proof. See [40, Lemma 3.6]. ■

As an application of Theorem 1.1.3 and Lemma 1.1.4, we obtain the following result.

?⟨prop.sm⟩?

Proposition 1.1.7 *Let X be a multivalued monotone vector field on M, $q \in M$, and $\lambda > 0$. Then, $X + \lambda \operatorname{grad} d_q^2$ is a strongly monotone vector field. Moreover, if X is maximal then $X + \lambda \operatorname{grad} d_q^2$ is also maximal.*

Proof. The first part is based on a combination of (1.9), (1.10), and [21, Proposition 3.2]. The second part is based on a straight combination of the convexity of d_q^2, Theorem 1.1.3, and Lemma 1.1.4. ■

1.2 Enlargement of monotone vector fields

⟨enlargement⟩

In this section, we extend the concept and some basic properties of enlargement of monotone vector fields from the Euclidean to the Hadamard setting.

⟨def.enl.X⟩

Definition 1.2.1 *Let X be a multivalued monotone vector field on M and $\epsilon \geq 0$. The enlargement of vector field $X^\epsilon : M \rightrightarrows TM$ associated to X is defined by*

$$X^\epsilon(p) := \left\{u \in T_pM \ : \ \left\langle P_{qp}^{-1}u - v, \exp_q^{-1} p\right\rangle \geq -\epsilon, \ q \in dom\, X, \ v \in X(q)\right\}, \quad p \in dom\, X.$$

In particular, when $M = \mathbb{R}^n$, Definition 1.2.1 retrieves the definition of enlargement of monotone operators introduced in [17]. It is worth noting that the definition of enlargement of monotone operators follows the same philosophy as that of the ϵ-subdifferential; see for example [13,34,35]. In other words, this important concept was introduced in order to provide more latitude and more robustness to some methods, including proximal and extragradient methods; see [16] and its reference therein. In the next proposition, it is shown that X^ϵ effectively constitutes an enlargement to X.

⟨prop.elem.ii⟩

Proposition 1.2.2 *Let X be a monotone vector field on M and $\epsilon \geq 0$. Then, $X \subset X^\epsilon$ and $\operatorname{dom} X \subset \operatorname{dom} X^\epsilon$. In particular, if $\operatorname{dom} X = M$ then $\operatorname{dom} X^\epsilon = \operatorname{dom} X$. Moreover, if X is maximal then $X^0 = X$.*

Proof. Consider $\epsilon \geq 0$. Since X is monotone, the first part of the proposition is based on (1.9) and Definition 1.2.1. Thus, using that $\operatorname{dom} X = M$, we conclude that $\operatorname{dom} X^\epsilon = \operatorname{dom} X$. The proof of the last part is based in Definition 1.2.1 and the maximality of X, and considering that $X \subset X^0$. ■

Now, we present a specific example to show how large the enlargement can become; see too [17] for other examples on linear spaces.

⟨ex:xe⟩ **Example 1.2.3** Let $\epsilon \geq 0$ and $\bar{p} \in M$. Define the closed ball at the origin 0_{T_pM} of T_pM and radius $2\sqrt{2\epsilon}$ by

$$B\left[0_{T_pM},\ 2\sqrt{2\epsilon}\right] := \left\{w \in T_pM \ : \ \| w \| \leq 2\sqrt{2\epsilon}\right\}.$$

Denote by $\partial^\epsilon d^2_{\bar{p}}(\cdot)$, the enlargement of the vector field $\partial d^2_{\bar{p}}(\cdot) = \{\operatorname{grad} d^2_{\bar{p}}(\cdot)\}$ defined in (1.1). We claim that the following inclusion holds

$$\partial d^2_{\bar{p}}(p) + B\left[0_{T_pM},\ 2\sqrt{2\epsilon}\right] \subseteq \partial^\epsilon d^2_{\bar{p}}(p), \qquad p \in M.$$

Indeed, first note that from (1.1) we have $\partial d^2_{\bar{p}}(q) = \{-2\exp^{-1}_q \bar{p}\}$, for each $q \in M$. Owing to $\operatorname{dom} \partial d^2_{\bar{p}} = M$, the definition of $\partial^\epsilon d^2_{\bar{p}}$ implies that

$$\partial^\epsilon d^2_{\bar{p}}(p) = \left\{u \in T_pM \ : \ -\left\langle u,\, \exp^{-1}_p q\right\rangle + \left\langle 2\exp^{-1}_q \bar{p},\, \exp^{-1}_q p\right\rangle \geq -\epsilon,\ q \in M\right\}, \quad p \in M. \tag{1.13}$$

ex.1.i

Next, we prove the auxiliary result $\{-2\exp^{-1}_p \bar{p}\} + A(p) \subset \partial^\epsilon d^2_{\bar{p}}(p)$ for each $p \in M$, where

$$A(p) = \left\{w \in T_pM \ : \ 0 \geq -2d^2(p,q) + \|w\| d(p,q) - \epsilon,\ q \in M\right\}, \quad p \in M. \tag{1.14}$$

eq:ap

First, note that by using (1.3), we obtain the following inequality

$$2\left[\langle \exp^{-1}_p \bar{p},\, \exp^{-1}_p q\rangle + \langle \exp^{-1}_q \bar{p},\, \exp^{-1}_q p\rangle - d^2(p,q)\right] \geq 0, \quad p, q \in M.$$

Consider $w \in A(p)$. Since $\langle w, \exp^{-1}_p q\rangle \leq \|w\| d(p,q)$, for all $w \in A(p)$ and $p, q \in M$, combining (1.14) with the last inequality yields

$$2\left[\langle \exp^{-1}_p \bar{p}, \exp^{-1}_p q\rangle + \langle \exp^{-1}_q \bar{p}, \exp^{-1}_q p\rangle - d^2(p,q)\right] \geq -2d^2(p,q) + \langle w, \exp^{-1}_p q\rangle - \epsilon, \quad p, q \in M.$$

Through simple algebraic manipulations in the last inequality, we obtain that it is equivalent to the following

$$-\langle -2\exp^{-1}_p \bar{p} + w,\, \exp^{-1}_p q\rangle + \langle 2\exp^{-1}_q \bar{p},\, \exp^{-1}_q p\rangle \geq -\epsilon, \quad p, q \in M,$$

which, from (1.13), allows us to conclude that $-2\exp^{-1}_p \bar{p} + w \in \partial^\epsilon d^2_{\bar{p}}(p)$, for all $w \in A(p)$ and $p \in M$. Thus, the auxiliary result is proved. Finally, note that $w \in A(p)$ if, and only if, there holds $\|w\|^2 - 8\epsilon < 0$, or equivalently, $\|w\| < 2\sqrt{2\epsilon}$. Therefore, $A(p) = B\left[0_{T_pM},\ 2\sqrt{2\epsilon}\right]$ and, because $\partial d^2_{\bar{p}}(p) + A(p) \subset \partial^\epsilon d^2_{\bar{p}}(p)$ for each $p \in M$, the proof of the claim is completed.

Remark 1.2.4 If M has zero curvature then (1.3) holds as the equality. Therefore, in Example 1.2.3, we can prove that the inequality holds as equality, namely, $\partial d^2_{\bar{p}}(p) + B\left[0_{T_pM},\ 2\sqrt{2\epsilon}\right] = \partial^\epsilon d^2_{\bar{p}}(p)$, for all $p \in M$.

We proceed with some basic properties of the enlargement of multivalued monotone vector fields, which are extensions to the Riemannian context of the corresponding one of linear setting; see [17].

⟨prop.elem.X⟩

Proposition 1.2.5 *Let X, X_1, and X_2 be multivalued monotone vector fields on M and $\epsilon, \epsilon_1, \epsilon_2 \geq 0$. Then the following statements hold:*

i) *If $\epsilon_1 \geq \epsilon_2 \geq 0$, then $X^{\epsilon_2} \subset X^{\epsilon_1}$;*

ii) $X_1^{\epsilon_1} + X_2^{\epsilon_2} \subset (X_1 + X_2)^{\epsilon_1+\epsilon_2}$;

iii) *$X^{\epsilon}(p)$ is closed and convex for all $p \in M$;*

iv) *$\alpha X^{\epsilon} = (\alpha X)^{\alpha\epsilon}$ for all $\alpha \geq 0$;*

v) *$\alpha X_1^{\epsilon} + (1-\alpha)X_2^{\epsilon} \subset (\alpha X_1 + (1-\alpha)X_2)^{\epsilon}$ for all $\alpha \in [0,1]$;*

vi) *If $E \subset \mathbb{R}_+$, then $\bigcap_{\epsilon \in E} X^{\epsilon} = X^{\overline{\epsilon}}$ with $\overline{\epsilon} = \inf E$.*

Proof. The proof is an immediate consequence of Definition 1.2.1. ∎

rop.conv.alg.⟩

Proposition 1.2.6 *Let X be a multivalued monotone vector field on M, $\{\epsilon^k\}$ be a sequence of positive numbers, and $\{(p^k, u^k)\}$ be a sequence in TM. If $\overline{\epsilon} = \lim_{k\to\infty} \epsilon^k$, $\overline{p} = \lim_{k\to\infty} p^k$, $\overline{u} = \lim_{k\to\infty} u^k$, and $u^k \in X^{\epsilon_k}(p^k)$ for all k, then $\overline{u} \in X^{\overline{\epsilon}}(\overline{p})$.*

Proof. Since $u^k \in X^{\epsilon_k}(p^k)$ for all k, then from Definition 1.2.1 we have

$$-\langle u^k, \exp^{-1}_{p^k} q\rangle + \langle -v, \exp^{-1}_q p^k\rangle \geq -\epsilon_k, \qquad q \in \operatorname{dom} X, \quad v \in X(q).$$

Taking limits in the last inequality, as k goes to infinity, we conclude that

$$-\langle \overline{u}, \exp^{-1}_{\overline{p}} q\rangle + \langle -v, \exp^{-1}_q \overline{p}\rangle \geq -\overline{\epsilon}, \qquad q \in \operatorname{dom} X, \quad v \in X(q).$$

Therefore, using again Definition 1.2.1, we obtain the desired result. ∎

op.boun.boun.⟩

Proposition 1.2.7 *If X is maximal monotone and $\operatorname{dom} X = M$, then X^{ϵ} is bounded on bounded sets, for all $\epsilon \geq 0$.*

Proof. Since X is monotone and $\operatorname{dom} X = M$, Proposition 1.2.2 implies that $\operatorname{dom} X^{\epsilon} = M$. Consider $V \subset M = \operatorname{int}(\operatorname{dom} X^{\epsilon})$ is a bounded set. Note that $\overline{V} \subset \operatorname{int}(\operatorname{dom} X^{\epsilon})$. Let $r > 0$ and the set be defined as $V_r = \{p \in M \ : \ d(p, V) \leq r\}$. Considering that $\operatorname{dom} X = M$, then $V_r \subset \operatorname{dom} X$. Moreover, since both sets V and V_r are bounded, Proposition 1.1.6 implies that $m_X(V) < +\infty$ and $m_X(V_r) < +\infty$. We prove that

$$m_{X^{\epsilon}}(V) \leq \frac{\epsilon}{r} + m_X(V_r) + 2m_X(V). \tag{1.15}$$ eq:lxve

Consider $p \in V$, $u \in X^\epsilon(p)$. Thus, for all $v \in X(q)$, Definition 1.2.1 implies that

$$-\epsilon \leq -\langle u, \exp_p^{-1} q\rangle - \langle v, \exp_q^{-1} p\rangle.$$

Let $\hat{u} \in X(p)$. For $\hat{u} \neq u$ define $q = \exp_p w$, where $w = (r/\|u - \hat{u}\|)(u - \hat{u})$. Thus, the last inequality becomes

$$-\epsilon \leq -\|u - \hat{u}\|r - \langle \hat{u}, \exp_p^{-1} q\rangle - \langle v, \exp_q^{-1} p\rangle.$$

Since the parallel transport is isometric, we conclude from the last inequality that

$$-\epsilon \leq -\|u - \hat{u}\|r + \|\exp_q^{-1} p\|\|P_{qp}^{-1}\hat{u} - v\|.$$

Since $r = \|\exp_q^{-1} p\|$, using the triangle inequality and that the parallel transport is isometric, along with some manipulation in the last inequality, we obtain $\|u - \hat{u}\| \leq \epsilon/r + \|\hat{u}\| + \|v\|$. Hence, considering that $\|u\| \leq \|u - \hat{u}\| + \|\hat{u}\|$, we obtain

$$\|u\| \leq \frac{\epsilon}{r} + 2\|\hat{u}\| + \|v\|.$$

Note that the last inequality also holds for $u = \hat{u}$. Since $\|\exp_q^{-1} p\| = r$ and $p \in V$, we have $q \in V_r$. Thus, $\|\hat{u}\| \leq m_X(\Omega)$ and $\|v\| \leq m_X(\Omega_r)$, which implies that

$$\|u\| \leq \frac{\epsilon}{r} + m_X(\Omega_r) + 2m_X(\Omega).$$

Since u is an arbitrary element of $X^\epsilon(\Omega)$, the inequality in (1.15) follows, and the proof is concluded. ∎

In the next definition we extend the notion of lower semicontinuity of a multivalued operator, which has been introduced in [17], to a vector field.

Definition 1.2.8 *A multivalued vector field $Y : M \rightrightarrows TM$ is said to be lower semicontinuous at $\bar{p} \in \mathrm{dom}Y$ if, for each sequence $\{p^k\} \subset \mathrm{dom}Y$ such that $\lim_{k\to+\infty} p^k = \bar{p}$ and each $\bar{u} \in Y(\bar{p})$, there exists a sequence $\{w^k\}$ such that $w^k \in Y(p^k)$ and $\lim_{k\to\infty} P_{p^k\bar{p}} w^k = \bar{u}$.*

The next result is a generalization of Theorem 4.1 of [36], it will play an important role in the convergence analysis of the extragradient method in Chapter 3.

⟨Theoremlsc⟩ **Theorem 1.2.9** *Let $X : M \rightrightarrows TM$ be a maximal monotone vector field and $\epsilon > 0$. If $domX = M$ then X^ϵ is lower semicontinuous.*

Proof. Since $\mathrm{dom}X = M$, Proposition 1.2.2 implies $domX = domX^\epsilon$. Take $\{p^k\} \subset M$ such that $\lim_{k\to+\infty} p^k = \bar{p}$ and $\bar{u} \in X^\epsilon(\bar{p})$. First, we are going to prove that the following statements there hold:

(i) For each $0 < \theta < 1$ and $u^k \in X(p^k)$, there exists $k_0 \in \mathbb{N}$ such that $(1-\theta)P_{\bar{p}p^k}\bar{u} + \theta u^k \in X^\epsilon(p^k)$, for all $k > k_0$;

(ii) Take $\nu > 0$. Then, there exist $k_0 \in \mathbb{N}$ and $v_k \in X^\epsilon(p^k)$ such that $\| \bar{u} - P_{p^k\bar{p}}v_k \| \leq \nu$, for all $k > k_0$.

For proving (i), take $q \in M$ and $v \in X(q)$. Then, simple algebraic manipulations yield

$$\left\langle (1-\theta)P_{\bar{p}p^k}\bar{u} + \theta u^k - P_{qp^k}v, \exp^{-1}_{p^k} q \right\rangle = (1-\theta)\left\langle P_{\bar{p}p^k}\bar{u} - P_{qp^k}v, \exp^{-1}_{p^k} q \right\rangle + \theta \left\langle u^k - P_{qp^k}v, \exp^{-1}_{p^k} q \right\rangle.$$

Since X is monotone, the second term in the right hand said of the last inequality is positive. Thus,

$$\left\langle (1-\theta)P_{\bar{p}p^k}\bar{u} + \theta u^k - P_{qp^k}v, \exp^{-1}_{p^k} q \right\rangle \geq (1-\theta)\left\langle P_{\bar{p}p^k}\bar{u} - P_{qp^k}v, \exp^{-1}_{p^k} q \right\rangle. \tag{1.16}$$ lemma_lsc_i

On the other hand, considering that $\lim_{k\to\infty}\langle P_{\bar{p}p^k}\bar{u} - P_{qp^k}v, \exp^{-1}_{p^k} q\rangle = \langle \bar{u} - P_{q\bar{p}}v, \exp^{-1}_{\bar{p}} q\rangle \geq -\epsilon$. Then, for all $\delta > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$\langle P_{\bar{p}p^k}\bar{u} - P_{qp^k}v, \exp^{-1}_{p^k} q\rangle \geq -\epsilon - \delta, \qquad k > k_0. \tag{1.17}$$ lemma_lsc_ii

Combining (1.16) and (1.17) and taking $\delta = \theta\epsilon/(1-\theta)$ we conclude that

$$\langle (1-\theta)P_{\bar{p}p^k}\bar{u} + \theta u^k - P_{qp^k}v, \exp^{-1}_{p^k} q\rangle \geq -\epsilon, \qquad k > k_0,$$

which proof the item (i).

For proving the item (ii), take $\eta > 0$ and consider the following auxiliary constants:

$$\sigma := \sup\left\{\| u \| : u \in X^\epsilon(B(\bar{p},\eta))\right\}, \qquad \gamma := \min\{(\epsilon/2\sigma), \eta\}, \qquad 0 < \mu < \min\{1, (\nu/2\sigma)\}.$$

Take any $u^k \in X(p^k)$. Applying item (i) with $\theta = \mu$, we conclude that there exists $k_0 \in \mathbb{N}$ such that $(1-\mu)P_{\bar{p}p^k}\bar{u} + \mu u^k \in X^\epsilon(p^k)$, for all $k \geq k_0$. We are going to prove that, taking $v_k = (1-\mu)P_{\bar{p}p^k}\bar{u} + \mu u^k$, we have $\| \bar{u} - v_k \| \leq \nu$, for all $k \geq k_0$. First note that, some manipulation and taking into account that the parallel transport is an isometry we have

$$\| \bar{u} - v_k \| = \mu \| \bar{u} - P_{p^k\bar{p}}u^k \| \leq \mu(\| \bar{u} \| + \| u^k \|), \qquad k \geq k_0. \tag{1.18}$$ est. bar{u}-u

Since $\lim_{k\to+\infty} p^k = \bar{p}$, there exist k_0 such that $p^k \in B(\bar{p},\gamma)$, for all $k \geq k_0$. Thus, taking into account that $u^k \in X(p^k) \subset X^\epsilon(p^k)$ and $B(\bar{p},\gamma) \subset B(\bar{p},\eta)$, the definition of σ gives $\| u^k \| \leq \sigma$, for all $k \geq k_0$. Due to $\bar{u} \in X^\epsilon(\bar{p})$ we also have $\| \bar{u} \| \leq \sigma$. Therefore, using (1.18) and the definition of μ we obtain

$$\| \bar{u} - v_k \| \leq 2\sigma\mu \leq \nu, \qquad k \geq k_0, \tag{1.19}$$ {?}

and the proof of item (ii) is proved.

Finally, we define the sequence $\{w^k\}$ as follows $w^k := \text{argmin}\{\|\bar{u} - P_{p^k \bar{p}} u\| \; : \; u \in X^{\epsilon}(p^k)\}$ for each k. Since, for each k the set $X^{\epsilon}(p^k)$ is closed and convex, the sequence $\{w^k\}$ is well defined. We claim that $\lim_{k\to\infty} P_{p^k \bar{p}} w^k = \bar{u}$. Otherwise, there exists $\{p^{k_j}\}$ a subsequence of $\{p^k\}$ and some $\nu > 0$ such that $\| \bar{u} - P_{p^{k_j} \bar{p}} w^{k_j} \| > \nu$ for all j. Definition of the sequence $\{w^k\}$ implies that $\| \bar{u} - P_{p^{k_j} \bar{p}} u \| > \nu$ for all $u \in X^{\epsilon}(p^{k_j})$ and all j. On the other hand, considering that $\lim_{k_j \to +\infty} p^{k_j} = \bar{p}$, $\bar{u} \in X^{\epsilon}(\bar{p})$ and the item (ii) holds, for all $\nu > 0$, we have a contraction. Therefore, the claim is proven and the proof is concluded. ■

Remark 1.2.10 The importance of this last proposition resides in the fact that, even in Euclidean spaces, a maximal monotone operator is not always lower semicontinuous; hence the need to introduce the enlargement in the algorithm proposed as an alternative to ensure the convergence of method without additional hypothesis on the X operator; see [36, Section 2] for more details.

Chapter 2

An Inexact Proximal Point Method for Variational Inequalities on Hadamard Manifolds

⟨chapter2⟩

The objective of this chapter is to present an inexact proximal point method for variational inequalities in Hadamard manifolds and to study its convergence properties. As an application, we obtain an inexact proximal point method for constrained optimization problems in Hadamard manifolds. It is worth mentioning that the concept of enlargement of monotone operators in linear spaces has been successfully employed for a wide range of purposes; see [16] and its reference therein. To the best of our knowledge, this is the first time that the inexact proximal point method for variational inequalities using the concept of enlargement is studied in the Riemannian setting. Finally, we also state that the proposed method has two important particular instances, namely, the methods (5.1) of [42] and (4.3) of [40].

Based on the concept of enlargement studied in Section (1.2), we introduce an inexact proximal point method for variational inequalities in Hadamard manifolds. It is worth noting that, the proximal point method on Riemannian manifolds was first introduced by O. P. Ferreira et al. in [29]. It is relevant to mention that the idea of to use the structure of Hadamard manifolds for optimization methods didn't exists, [29] was the first work to take into account that the curvature of manifold Riemannian plays a crucial role in the convergence analysis of the method. Since then it has become quite usual to work on Hadamard manifolds.

Variational inequalities in Hadamard Manifolds was first introduced in [47], for single-valued vector fields on Hadamard manifolds, and in [42] for multivalued vector fields in Riemannian manifolds. The definition of the variational inequality for multivalued vector fields in Hadamard manifolds is:

Let $X : M \rightrightarrows TM$ be a multivalued vector field and $\Omega \subset M$ be a nonempty set. The *variational inequality* VIP(X,Ω) involves finding $p^* \in \Omega$ such that there exists $u \in X(p^*)$ satisfying

$$\langle u, \exp^{-1}_{p^*} q\rangle \geq 0, \qquad q \in \Omega.$$

Using (1.6), i.e., the definition of normal cone to Ω, VIP(X,Ω) becomes the problem of finding an $p^* \in \Omega$ that satisfies the inclusion

$$0 \in X(p) + N_{\Omega}(p). \tag{2.1}$$

eq.vip

Remark 2.0.11 In particular, if $\Omega = M$, then $N_{\Omega}(p) = \{0\}$ and VIP (X,Ω) are problems with regard to finding $p^* \in \Omega$ such that $0 \in X(p^*)$.

2.1 An Inexact Proximal Point Method

⟨ppm⟩

Hereafter, $S(X, \Omega)$ denotes the *solution set of the inclusion* (2.1). We require the following three assumptions:

A1. $\operatorname{dom} X = M$ and Ω closed and convex;

A2. X is maximal monotone;

A3. $S(X, \Omega) \neq \varnothing$.

Consider $0 < \hat{\lambda} \leq \tilde{\lambda}$, a sequence $\{\lambda_k\} \subset \mathbb{R}$ such that $\hat{\lambda} \leq \lambda_k \leq \tilde{\lambda}$, and a sequence $\{\epsilon_k\} \subset \mathbb{R}_+$ such that $\sum_{k=0}^{\infty} \epsilon_k < \infty$. The *proximal point method* for VIP (X, Ω) is defined as follows: Given $p^0 \in \Omega$ take p^{k+1} such that

$$0 \in (X^{\epsilon_k} + N_{\Omega})(p^{k+1}) - 2\lambda_k \exp^{-1}_{p^{k+1}} p^k, \qquad k = 0, 1 \ldots. \tag{2.2}$$

eq.pk+1ii

Remark 2.1.1 Method (2.2) has many important particular instances. For example, in the case $\epsilon_k = 0$ for all k, we obtain method (5.1) of [42]. For $\Omega = M$ and $\epsilon_k = 0$ for all k, we obtain method (4.3) of [40]. For $M = \mathbb{R}^n$, we obtain method (23)-(25) of [17], where the Bregman distance is induced by the square of the Euclidean norm and $C = \mathbb{R}^n$. It is worth mentioning that an inexact proximal point method on Hadamard manifolds has already been studied before; see [2, 54, 59]. However, subproblem (2.2), which uses the enlargement X^{ϵ}, is considerably different from the subproblems defining the inexact proximal sequence in [2, 54, 59].

⟨ex.i⟩

Example 2.1.2 Let $\mathbb{H}^2 := \{x = (x_1, x_2) \in \mathbb{R}^2 \ : \ x_2 > 0, \}$ be the 2-dimensional hyperbolic space endowed with the Riemannian metric $g_{ij}(x_1, x_2) := \delta_{ij}/x_2^2$, for $i, j = 1, 2$. The curvature of $\mathbb{H}^2$ is $K = -1$, and the geodesics in $\mathbb{H}^2$ are semicircles centered on x_1-axis and vertical lines. Udriste discusses more details of this in [58].

The perpendicular from (x, y) to Oy is the geodesic $x^2 + y^2 = a^2$, $y > 0$. Let $f : \mathbb{H}^2 \to \mathbb{R}$ s.t. $f(x, y) = \ln^2 \frac{x+a}{y}$, the square of the distance from $P_1 = (x, y) \in H$ to the vertical geodesic Oy, is a convex function; see pag. 87 of [58].

In view of Proposition 3.4 (i) of [21] and Theorem 1.1.3 we obtain that $\operatorname{grad} f$ is maximal monotone vector field. Thus, we can apply the algorithm (2.2), with $X = \operatorname{grad} f$, $M = \mathbb{H}^2$ and $\Omega = \{(x, y) \in \mathbb{R}^2 | 1 \leq x^2 + y^2 \leq 2\}$ for to solve VIP(X, Ω) (note that $S(X, \Omega) = \{(0, y) \in \mathbb{R}^2 | 1 \leq y \leq 2\} \neq \varnothing$). On the other hand, we can not apply the classical methods for linear spaces, because, endowing $\mathbb{H}$ with the Euclidean metric, X is nonmonotone and Ω is nonconvex set.

⟨le:esvi⟩

Lemma 2.1.3 *For each $q \in M$ and $\lambda > 0$, the inclusion problem $0 \in (X + N_\Omega)(p) - 2\lambda \exp_p^{-1} q$, for $p \in M$, has a unique solution.*

Proof. Is sufficient to combine Corolary 3.14 of [42], Proposition (1.1.6) and Theorem 3.7 of [40]. ■

Now, we prove a convergence result for the proximal point method (2.2).

⟨conv.alg.ii⟩

Theorem 2.1.4 *Assume that* **A1-A3** *hold. Then, the sequence $\{p^k\}$ generated by (2.2) is well defined and converges to a point $p^* \in S(X, \Omega)$.*

Proof. Since $\operatorname{dom} X = M$, Proposition 1.2.2 and item (i) of Proposition 1.2.7 imply that $X(p) \subseteq X^{\epsilon_k}(p)$ for all $p \in M$ and $k = 0, 1, \ldots$. Hence, for proving the definition of the sequence $\{p^k\}$, it is sufficient to prove that the inclusion

$$0 \in (X + N_\Omega)(p) - 2\lambda_k \exp_p^{-1} p^k, \qquad p \in M,$$

has a solution, for each $k = 0, 1, \ldots$, which is a consequence of Lemma 2.1.3.

Now, we are going to prove the convergence of $\{p^k\}$ to a point $p^* \in S(X, \Omega)$. Using Proposition 1.2.2 we conclude that $N_\Omega \subset N_\Omega^0$. Thus, from item ii of Proposition 1.2.5, we have $X^{\epsilon_k} + N_\Omega \subset (X + N_\Omega)^{\epsilon_k}$, for all $k = 0, 1, \ldots$. Therefore, using (2.2), we obtain

$$2\lambda_k \exp_{p^{k+1}}^{-1} p^k \in (X + N_\Omega)^{\epsilon_k}(p^{k+1}), \qquad k = 0, 1, \ldots. \tag{2.3}$$

eq:icte

Considering that $P_{qp^{k+1}}^{-1} \exp_q^{-1} p^{k+1} = -\exp_{p^{k+1}}^{-1} q$ and the parallel transport being isometric, the last inclusion together with Definition 1.2.1 yields

$$-2\lambda_k \left\langle \exp_{p^{k+1}}^{-1} p^k, \exp_{p^{k+1}}^{-1} q \right\rangle + \left\langle v, -\exp_q^{-1} p^{k+1} \right\rangle \geq -\epsilon_k, \quad q \in \Omega,\ v \in (X + N_\Omega)(q), \quad k = 0, 1, \ldots.$$

In particular, if $q \in S(X, \Omega)$, then $0 \in X + N_\Omega(q)$ and the last inequality becomes

$$-2\lambda_k \left\langle \exp^{-1}_{p^{k+1}} p^k, \exp^{-1}_{p^{k+1}} q \right\rangle \geq -\epsilon_k, \quad q \in S(X, \Omega), \qquad k = 0, 1, \ldots.$$

Using the last inequality and (1.2) with $p_1 = p^k$, $p_2 = q$, and $p_3 = p^{k+1}$, along with some algebraic calculations, we obtain

$$-\frac{\epsilon_k}{2\lambda_k} \leq d^2(q, p^k) - d^2(p^k, p^{k+1}) - d^2(q, p^{k+1}), \qquad q \in S(X, \Omega), \qquad k = 0, 1, \ldots. \tag{2.4}$$

eq.the.ii.X

Since $0 < \hat{\lambda} \leq \lambda_k$, the last inequality gives

$$d^2(q, p^{k+1}) \leq d^2(q, p^k) + \frac{\epsilon_k}{\hat{\lambda}}, \qquad q \in S(X, \Omega), \qquad k = 0, 1, \ldots. \tag{2.5}$$

?eq.fejer.X?

Because $\sum_{k=0}^{\infty} \epsilon_k < \infty$ and $S(X, \Omega) \neq \varnothing$, the last inequality implies that $\{p^k\}$ is quasi-Fejér convergent to $S(X, \Omega)$. From Proposition 1.0.3, for concluding the proof, it is sufficient to prove that there exists an accumulation point $\bar{p}$ of $\{p^k\}$ belonging to $S(X, \Omega)$. Since $\{p^k\}$ is quasi-Fejér convergent to $S(X, \Omega)$, Proposition 1.0.3 implies that $\{p^k\}$ is bounded. Consider $\bar{p}$ and $\{p^{n_k}\}$, an accumulation point and a subsequence of $\{p^k\}$, respectively, such that $\bar{p} = \lim_{k\to\infty} p^{n_k}$. On the other hand, since $0 < \hat{\lambda} \leq \lambda_k$ and $\sum_{k=0}^{\infty} \epsilon_k < \infty$, the inequality in (2.4) implies that $\lim_{k\to\infty} d(p^k, p^{k+1}) = 0$. Thus, $\lim_{k\to\infty} \exp^{-1}_{p^{n_k+1}} p^{n_k} = 0$ and $\lim_{k\to\infty} p^{n_k+1} = \bar{p}$. Now, using (2.3), we have

$$2\lambda_{n_k} \exp^{-1}_{p^{n_k+1}} p^{n_k} \in (X + N_\Omega)^{\epsilon_{n_k}}(p^{n_k+1}), \qquad k = 0, 1, \ldots.$$

Therefore, if k tends to infinity in the last inclusion, using Proposition 1.2.6, Lemma 1.1.5, Proposition 1.2.2, and considering that $\{\lambda_k\}$ is bounded we obtain $0 \in (X + N_\Omega)(\bar{p})$, which implies that $\bar{p} \in S(X, \Omega)$ and the proof is concluded. ■

Remark 2.1.5 In [59], an inexact proximal point method for constrained optimization problems on Hadamard manifolds is presented, where the condition guaranteeing convergence is weaker than the condition in the above theorem. On the other hand, the enlargement X^ϵ is an (outer) approximation to X. Consequently, even in the linear setting, the proximal subproblem using the enlargement has the advantage of providing more latitude and more robustness to the methods used for solving it; see [16, 17].

2.2 An Inexact Proximal Point Method for Optimization

In this section, we apply the results of the section 2.1 to obtain an inexact proximal point method for the constrained optimization problems in Hadamard manifolds. Throughout this section, we assume that $f : M \to \mathbb{R}$ is a convex function. The *enlargement of the subdifferential of* f, denoted by $\partial^\epsilon f : M \rightrightarrows TM$, is defined by

$$\partial^\epsilon f(p) := \left\{u \in T_pM \ : \ \left\langle \mathrm{P}_{qp}^{-1} u - v, \exp_q^{-1} p\right\rangle \geq -\epsilon, \ q \in M, \ v \in \partial f(q)\right\}, \qquad \epsilon \geq 0.$$

and we denote the ϵ-*subdifferential* of f by $\partial_\epsilon f : M \rightrightarrows TM$, which is given by

$$\partial_\epsilon f(p) := \left\{u \in T_pM \ : \ f(q) \geq f(p) + \left\langle u, \exp_p^{-1} q\right\rangle - \epsilon, \ q \in M\right\}, \qquad \epsilon \geq 0.$$

The next example shows how big the ϵ-*subdifferential* of the squared distance function can become.

⟨ex:xesg⟩

Example 2.2.1 Let $\epsilon \geq 0$ and $\bar{p} \in M$. Define the closed ball at the origin 0_{T_pM} of T_pM and radius $2\sqrt{\epsilon}$ by

$$B\left[0_{T_pM}, \ 2\sqrt{\epsilon}\right] := \left\{w \in T_pM \ : \ \| \, w \, \| \leq 2\sqrt{\epsilon}\right\}.$$

Denote the ϵ-subdifferential of $d_{\bar{p}}^2$ by $\partial_\epsilon d_{\bar{p}}^2$. Thus, we claim that the following inclusion holds

$$\partial d_{\bar{p}}^2(p) + B\left[0_{T_pM}, \ 2\sqrt{\epsilon}\right] \subseteq \partial_\epsilon d_{\bar{p}}^2(p), \qquad p \in M.$$

Indeed, first note that (1.1) implies that $\partial d_{\bar{p}}^2(q) = \{-2\exp_q^{-1} \bar{p}\}$, for each $q \in M$. Owing to $\operatorname{dom} \partial d_{\bar{p}}^2 = M$, the definition of $\partial_\epsilon d_{\bar{p}}^2$ implies that

$$\partial_\epsilon d_{\bar{p}}^2(p) = \left\{u \in T_pM \ : \ d^2(\bar{p}, q) \geq d^2(\bar{p}, p) + \left\langle u, \exp_p^{-1} q\right\rangle - \epsilon, \ q \in M\right\}, \quad p \in M. \qquad (2.6)$$ ex.1.isg

Now, we prove the auxiliary result $\{-2\exp_p^{-1} \bar{p}\} + B(p) \subset \partial_\epsilon d_{\bar{p}}^2(p)$ for each $p \in M$, where

$$B(p) = \left\{w \in T_pM \ : \ 0 \geq -d^2(p, q) + \|w\| d(p, q) - \epsilon, \ q \in M\right\}, \quad p \in M. \qquad (2.7)$$ eq:apsg

First, note that by using the inequality in (1.2), we obtain the following inequality

$$d^2(\bar{p}, q) - d^2(\bar{p}, p) - d^2(p, q) + 2\left\langle \exp_p^{-1} \bar{p}, \exp_p^{-1} q\right\rangle \geq 0, \quad p, q \in M.$$

Consider $w \in B(p)$. Since $\langle w, \exp_p^{-1} q\rangle \leq \|w\| d(p, q)$, for all $w \in B(p)$ and $p, q \in M$, combining (2.7) with the last inequality yields

$$d^2(\bar{p}, q) - d^2(\bar{p}, p) - d^2(p, q) + 2\left\langle \exp_p^{-1} \bar{p}, \exp_p^{-1} q\right\rangle \geq -d^2(p, q) + \left\langle w, \exp_p^{-1} q\right\rangle - \epsilon, \quad p, q \in M.$$

Simple algebraic manipulations in the latest inequality show that the latter is equivalent to the following

$$d^2(\bar{p}, q) \geq d^2(\bar{p}, p) + \left\langle -2\exp_p^{-1} \bar{p} + w, \exp_p^{-1} q\right\rangle - \epsilon, \quad p, q \in M,$$

which, from (2.6), allows us to conclude that $-2\exp_p^{-1} \bar{p} + w \in \partial_\epsilon d_{\bar{p}}^2(p)$, for all $w \in B(p)$ and $p \in M$. Thus, the auxiliary result is proved. Finally, note that $w \in B(p)$ if, and only if, there holds $\|w\|^2 - 4\epsilon < 0$, or equivalently, $\|w\| < 2\sqrt{\epsilon}$. Therefore, $B(p) = B\left[0_{T_pM}, \ 2\sqrt{\epsilon}\right]$ and, because $\partial d_{\bar{p}}^2(p) + B(p) \subset \partial_\epsilon d_{\bar{p}}^2(p)$ for each $p \in M$, the proof of the claim is completed.

The next proposition shows that the enlargement of the subdifferential of f is bigger than its ϵ-subdifferential.

prop.epsilon⟩?

Proposition 2.2.2 *For each $p \in M$, there holds $\partial_\epsilon f(p) \subseteq \partial^\epsilon f(p)$.*

Proof. Consider $u \in \partial_\epsilon f(p)$, $q \in M$, and $v \in \partial f(q)$. From the definitions of $\partial f(q)$ and $\partial_\epsilon f(p)$, we have

$$f(p) \geq f(q) + \langle v, \exp_q^{-1} p\rangle, \qquad\qquad f(q) \geq f(p) + \langle u, \exp_p^{-1} q\rangle - \epsilon,$$

respectively. Combining the last two inequalities, we conclude that $0 \geq \langle v, \exp_q^{-1} p\rangle + \langle u, \exp_p^{-1} q\rangle - \epsilon$. Since the parallel transport is isometric and $\mathrm{P}_{qp}^{-1} \exp_p^{-1} q = -\exp_q^{-1} p$, the last inequality becomes

$$0 \geq \langle v,\, \exp_q^{-1} p\rangle + \langle \mathrm{P}_{qp}^{-1} u,\, -\exp_q^{-1} p\rangle - \epsilon.$$

Thus, the last inequality and the definition of $\partial^\epsilon f(p)$ imply that $u \in \partial^\epsilon f(p)$, which complete the proof. ■

Remark 2.2.3 Note that if M has zero curvature then inequality (1.2) holds as an equality. Therefore, in example 2.2.1, it can be proved that in fact $\partial d_{\bar{p}}^2(p) + B\left[0_{T_pM},\ 2\sqrt{\epsilon}\right] = \partial_\epsilon d_{\bar{p}}^2(p)$, $p \in M$. Moreover, it can be also proved that the inclusion of $\partial_\epsilon d_{\bar{p}}^2(p) \subset \partial^\epsilon d_{\bar{p}}^2(p)$ is strict, for all $p \in M$; see Example 1.2.3.

Let $\Omega \subset M$. The *constrained optimization problem* consists of

$$\min\ f(p), \qquad \text{subject to} \quad p \in \Omega. \tag{2.8}$$

eq.cop

Let δ_Ω be the indicate function, defined by $\delta_\Omega(p) = 0$, if $p \in \Omega$ and $\delta_\Omega(p) = +\infty$ otherwise. Problem (2.8) is equivalent to

$$\min\ (f + \delta_\Omega)(p), \qquad \text{subject to} \quad p \in M.$$

Hereafter, let $\Omega \subset M$ be a closed and convex set and $S(f, \Omega)$ be the solution set of (2.8).

?⟨th:copoc⟩?

Theorem 2.2.4 *There holds $\partial(f + \delta_\Omega)(p) = \partial f(p) + N_\Omega(p)$, for each $p \in \Omega$. Moreover, $p^* \in S(f, \Omega)$ if, and only if, $0 \in \partial f(p^*) + N_\Omega(p^*)$.*

Proof. The first part was proved in [40, Proposition 5.4]. To prove the second part, first we use the convexity of Ω and f to conclude that $f + \delta_\Omega$ is also convex, and then use the first part to obtain the result. ■

Consider $0 < \hat{\lambda} \leq \tilde{\lambda}$, a sequence $\{\lambda_k\} \subset \mathbb{R}$ such that $\hat{\lambda} \leq \lambda_k \leq \tilde{\lambda}$ and a sequence $\{\epsilon_k\} \subset \mathbb{R}_{++}$ such that $\sum_{k=0}^{\infty} \epsilon_k < \infty$. The *inexact proximal point method for the constrained optimization problem* in (2.8) is defined as follows: Given $p^0 \in \Omega$ consider p^{k+1} such that

$$0 \in (\partial^{\epsilon_k} f + N_\Omega)(p^{k+1}) - 2\lambda_k \exp_{p^{k+1}}^{-1} x^k, \qquad k = 0, 1, \ldots. \tag{2.9}$$

eq.pia.iif

Remark 2.2.5 For $\epsilon_k = 0$, the above method generalizes method (5.15) of Chong Li et. al. [40] and, for $\epsilon_k = 0$ and $\Omega = M$, we obtain the method proposed by Ferreira and Oliveira [29].

?⟨conv.alg.⟩?

Theorem 2.2.6 *Assume that* $S(f, \Omega) \neq \varnothing$. *Then, the sequence* $\{p^k\}$ *generated by* (2.9) *is well defined and converges to a point* $p^* \in S(f, \Omega)$.

Proof. Since $\mathrm{dom} f = M$, Theorem 1.1.3 implies that ∂f is maximal monotone. Therefore, considering that $N_\Omega = \partial \delta_\Omega$, the result follows directly from Theorem 2.1.4 with $X = \partial f$. ■

Chapter 3

An Extragradient-Type Algorithm for Variational Inequality Problem on Hadamard Manifolds

⟨chapter3⟩

The objective of this chapter is to present an extragradient algorithm for variational inequalities in Hadamard manifolds and to study its convergence properties. In order to present our method, we utilize the concept of enlargement of monotone operators, introduced by [17] in Euclidean spaces and generalized by [7] from a linear setting to the Riemannian context; see also [15]. It is worth mentioning that the concept of enlargement of monotone operators in linear spaces has been successfully employed for a wide range of purposes; see [16] and its reference therein. Finally, we also state that the proposed method has two important particular instances, namely, the methods (3.1) of [36] and (4.1) of [53].

3.1 An Extragradient-type Algorithm

?⟨section3⟩?

In this section, we introduce an extragradient-type algorithm for variational inequalities in Hadamard manifolds.

ion_condition⟩

Lemma 3.1.1 *The following statements are equivalent:*

i) p^* *is a solution of* VIP*(X,Ω);*

ii) There exists $u^* \in X(p^*)$ *such that* $p^* = P_\Omega(\exp_{p^*}(-\alpha u^*))$, *for some* $\alpha > 0$.

Proof. It follows from (1.4). ■

We need the following three assumptions:

A1. $\operatorname{dom} X = M$ and $\Omega \subset M$ is closed and convex;

A2. X is maximal monotone;

A3. $S^*(X, \Omega) \neq \varnothing$.

We also need the following assumption, which plays an important role in the convergence analysis of our extragradient algorithm in Hadamard manifolds.

A4. For each $y \in M$ and $v \in T_yM$ the following set is convex

$$S := \left\{x \in M : \langle v, \exp_y^{-1} x\rangle \leq 0\right\}. \qquad (3.1)$$ def.S

Remark 3.1.2 In [30], it was shown that for Hadamard manifolds with constant curvature the set S in (3.1) is convex. The set S in (3.1) plays an important role in the strategy of the method and has been studied in some papers, including [60]. It is worth to point out that, so far it not known if or not (3.1) is convex in general Hadamard manifolds.

Next, we present an *extragradient-type algorithm for finding a solution of VIP*(X, Ω).

⟨Algorithm⟩ **Algorithm 3.1.3** Our algorithm requires six exogenous constants:

$$\epsilon > 0, \qquad 0 < \delta_- < \delta_+ < 1, \qquad 0 < \alpha_- < \alpha_+, \qquad 0 < \beta < 1, \qquad (3.2)$$ {?}

and two exogenous sequences $\{\alpha_k\}$ and $\{\beta_k\}$ satisfying the following conditions:

$$\alpha_k \in [\alpha_-,\ \alpha_+], \qquad \beta_k \in [\beta,\ 1], \qquad k = 0, 1, \ldots. \qquad (3.3)$$ {?}

1. Initialization: $p^0 \in \Omega, \qquad \epsilon_0 = \epsilon.$

2. Iterative step: Given p^k and ϵ_k,

(a). Selection of u^k: Find

$$u^k \in X^{\epsilon_k}(p^k), \qquad (3.4)$$ sel.uk.i

such that

$$\left\langle w, -\exp_{p^k}^{-1} P_\Omega(\exp_{p^k}(-\alpha_k u^k))\right\rangle \geq \frac{\delta_+}{\alpha_k} d^2\left(p^k, P_\Omega(\exp_{p^k}(-\alpha_k u^k))\right), \qquad w \in X^{\epsilon_k}(p^k). \quad (3.5)$$ sel.uk.ii

Define,

$$z^k := P_\Omega\left(\exp_{p^k}(-\alpha_k u^k)\right). \qquad (3.6)$$ sel.uk.iii

(b) Stopping criterion: If $p^k = z^k$, then stop. Otherwise,

(c) Selection of λ_k and v^k: Define $\gamma_k(t) := \exp_{p^k} t \exp^{-1}_{p^k} z^k$ and let

$$i(k) := \min\left\{ i \geq 0 \ : \ \exists\, v^{k,i} \in X(y^{k,i}),\ \left\langle v^{k,i}, \gamma'_k(2^{-i}\beta_k)\right\rangle \leq -\frac{\delta_-}{\alpha_k} d^2(p^k, z^k) \right\}, \qquad (3.7)$$ ik

where

$$y^{k,i} = \gamma_k\left(2^{-i}\beta_k\right). \qquad (3.8)$$ yki

Define

$$\lambda_k := 2^{-i(k)}\beta_k, \qquad y^k := \exp_{p^k} \lambda_k \exp^{-1}_{p^k} z^k, \qquad (3.9)$$ lambda_k

$$v^k := v^{k,i(k)}. \qquad (3.10)$$ v^k

(d) Definition of p^{k+1} and ϵ_{k+1}: Define

$$S_k := \{p \in M : \langle v^k, \exp^{-1}_{y^k} p\rangle \leq 0\}, \qquad q^k := P_{S_k}(p^k), \qquad (3.11)$$ S_k

$$p^{k+1} := P_\Omega(q^k), \qquad \epsilon_{k+1} := \min\left\{\epsilon_k,\ d^2(p^k, z^k)\right\}, \qquad (3.12)$$ def.p_k

and go to Iterative step.

Remark 3.1.4 If $M = \mathbb{R}^n$ the above algorithm retrieves one presented in [36] and if the field X is poit-to-point, continuous and monotone, and $\epsilon_k = 0$ for all k, we obtain the algorithm developed in [53]. Observe that we can apply the algorithm 3.1.3, with $X = \operatorname{grad} f$, $M = \mathbb{H}^2$ and $\Omega = \{(x, y) \in \mathbb{R}^2 | 1 \leq x^2 + y^2 \leq 2\}$ for to solve VIP(X, Ω) (note that $S(X, \Omega) = \{(0, y) \in \mathbb{R}^2 | 1 \leq y \leq 2\} \neq \varnothing$) (See example 2.1.2). However, we can not apply the classical methods for linear spaces, because, endowing $\mathbb{H}$ with the Euclidean metric, X is nonmonotone and Ω is nonconvex.

Next results establishes the well-definedness of the previous algorithm.

?⟨Boa_def.ii⟩? **Lemma 3.1.5** *Let $\{p^k\}$ be the sequence generated by the Algorithm 3.1.3. Then, there hold:*

(i) $p^k \in \Omega$, *for all k;*

(ii) *There exists u^k satisfying (3.4) and (3.5), for each k;*

(iii) *If* $p^k \neq z^k$, *then* $i(k)$ *is well defined.*

Proof. To prove item (i), it is sufficient to note that from initialization step we have $p^0 \in \Omega$ and that (3.12) implies $p^{k+1} \in \Omega$.

To prove item (ii) define the bifunction $f : T_pM \times T_pM \to \mathbb{R}$ by

$$f(v, w) = \langle \exp^{-1}_{p^k} z^k, v - w \rangle. \qquad (3.13)$$ eq:afim

Consider the equilibrium problem associated to f and $X^{\epsilon_k}(p^k)$, i.e., finding $u^k \in X^{\epsilon_k}(p^k)$ such that $f(u^k, w) \geq 0$ for all $w \in X^{\epsilon_k}(p^k)$. In view of item iii of Proposition 1.2.5 and Proposition 1.2.7 we have $X^{\epsilon_k}(p^k)$ compact and convex. Thus, since $X^{\epsilon_k}(p^k)$ is compact and convex and the function f in (3.13) satisfies all assumptions in [12, Basic Existence Theorem on page 3] (see also [6, Theorem 3.1]). Hence, we conclude that there is $u^k \in X^{\epsilon_k}(p^k)$ such that

$$\langle \exp^{-1}_{p^k} z^k, u^k \rangle \geq \langle \exp^{-1}_{p^k} z^k, w \rangle, \qquad w \in X^{\epsilon_k}(p^k). \qquad (3.14)$$ e.p.

On the other hand, taking $q = \exp_{p^k}(-\alpha_k u^k/\delta_+)$ and using inequality (1.3) with $p_1 = q$, $p_2 = p^k$ and $p_3 = z^k$ we obtain that

$$\langle \exp^{-1}_{p^k} q,\, \exp^{-1}_{p^k} z^k \rangle + \langle \exp^{-1}_{z^k} q,\, \exp^{-1}_{z^k} p^k \rangle \geq d^2(p^k, z^k).$$

Since $z^k = P_\Omega(q)$, we conclude from (1.4) and last inequality that

$$\langle \exp^{-1}_{p^k} q, \exp^{-1}_{p^k} z^k \rangle \geq d^2(p^k, z^k).$$

Considering that $q = \exp_{p^k}(-\alpha_k u^k/\delta_+)$, the latter inequality implies $\langle u^k, \exp^{-1}_{p^k} z^k \rangle \leq -\delta_+ d^2(p^k, z^k)/\alpha_k$. Therefore, combining this inequality with (3.14) the desired inequality follows.

For proving item (iii) we proceed by contradiction. Fix k and assume that, for each i, there holds

$$\left\langle v^{k,i},\ \gamma_k'(2^{-i}\beta_k) \right\rangle > -\frac{\delta_-}{\alpha_k} d^2(x^k, z^k), \qquad v^{k,i} \in X(y^{k,i}). \qquad (3.15)$$ itemiii

First note that, from (3.8) we have that $\{y^{k,i}\}$ belongs to the geodesic segment joining p_k to $\gamma_k(\beta_k)$. But this tell us that $\{y^{k,i}\}$ is a bounded sequence and, consequently, Proposition 1.1.6 implies that $\{X(y^{k,i})\}$ is also a bounded. Without loss of generality, we can assume that $\{v^{k,i}\}$ converges to $\bar{v}$. Letting i to infinity in (3.15) and taking into account that $\lim_{i\to\infty} v^{k,i} = \bar{v}$, $\gamma(0) = p^k$ and $\gamma'(0) = \exp^{-1}_{p^k} z^k$, we conclude that

$$\left\langle \bar{v},\ \exp^{-1}_{p^k} z^k) \right\rangle \geq -\frac{\delta_-}{\alpha_k} d^2(p^k, z^k). \qquad (3.16)$$ itemiii.2

On the other hand, since $\lim_{i\to\infty} y^{k,i} = p^k$, $\lim_{i\to\infty} v^{k,i} = \bar{v}$ and $v^{k,i} \in X(y^{k,i})$, using Proposition 1.1.2 we have $\bar{v} \in X(p^k)$. Thus, combining Propositions 1.2.2 and 1.2.5 (i) we obtain $\bar{v} \in X^{\epsilon}(p^k)$. Hence, using (3.6) and taking $w = \bar{v}$ the inequality (3.5) becomes

$$\langle \bar{v}, \exp^{-1}_{p^k} z^k) \rangle \leq -\frac{\delta_+}{\alpha_k} d^2(p^k, z^k). \tag{3.17}$$ itemiii.3

Since $0 < \delta^- < \delta^+$, the inequalities (3.16) and (3.17) imply that $d(p^k, z^k) = 0$, which is a contradiction to the fact that $p^k \neq z^k$. Therefore, $i(k)$ is well defined and the proof of the proposition is done. ■

From now on, $\{p^k\}$, $\{q^k\}$, $\{y^k\}$, $\{z^k\}$, $\{v^k\}$, $\{u^k\}$ and $\{\epsilon_k\}$ denote sequences generated by Algorithm 3.1.3. To prove the convergence of $\{p^k\}$ to a point of the solution set $S^*(X, \Omega)$, we need some preliminaries results.

⟨Fejer⟩ **Lemma 3.1.6** *The sequence $\{p^k\}$ is Fejér convergent to $S^*(X, \Omega)$ and $\lim_{k\to\infty} d(q^k, p^k) = 0$.*

Proof. We are going to show that, for all $p^* \in S^*(X, \Omega)$ there holds

$$d^2(p^*, p^{k+1}) \leq d^2(p^*, p^k) - d^2(q^k, p^k), \qquad k = 0, 1, \ldots. \tag{3.18}$$ eq.Fejer

Take $u^* \in X(p^*)$ such that $\langle u^*, \exp^{-1}_{p^*} q\rangle \geq 0$, for all $q \in \Omega$, and fix k. Due the monotonicity of X, we conclude that

$$\langle v^k, \exp^{-1}_{y^k} p^* \rangle \leq 0.$$

In view of (3.11), we obtain $p^* \in S_k$. On the other hand, applying (1.2) with $p_1 = p^*$, $p_2 = p_k$ and $p_3 = q_k$ we have

$$d^2(p^*, p^k) \geq d^2(p^*, q^k) + d^2(q^k, p^k) - 2\left\langle \exp^{-1}_{q^k} p^*, \exp^{-1}_{q^k} p^k \right\rangle.$$

Since $p^* \in S_k$ and $q^k = P_{S_k}(p^k)$, the last inequality implies that

$$d^2(p^*, p^k) \geq d^2(p^*, q^k) + d^2(q^k, p^k).$$

Analogously, applying (1.2) with $p_1 = p^*$, $p_2 = q_k$ and $p_3 = p_{k+1}$ and considering that $p^{k+1} := P_\Omega(q^k)$ and $p^* \in \Omega$, we conclude that

$$d^2(p^*, q^k) \geq d^2(p^*, p^{k+1}) + d^2(q^k, p^{k+1}).$$

Now, combining two last inequalities we obtain $d^2(p^*, p^k) \geq d^2(q^k, p^k) + d^2(p^*, p^{k+1}) + d^2(q^k, p^{k+1})$, which implies (3.18). In particular, (3.18) implies that $\{p^k\}$ is Féjer convergent to $S^*(X, \Omega)$ and $\{d(p^*, p^k)\}$ is noincresing and inferiorly limited. For concluding the proof, note that $\{d(p^*, p^k)\}$ converges. Thus, we have from (3.18) the desired result. ■

⟨convergence⟩

Lemma 3.1.7 *If the sequence $\{p^k\}$ is infinity then $\lim_{k\to\infty}\epsilon_k = 0$. Moreover, all accumulation points of $\{p^k\}$ belong to $S^*(X,\Omega)$.*

Proof. Suppose that the sequence $\{p^k\}$ is infinity, i.e., the algorithm does not stop. Thus, by the stopping criterion $d(p^k, z^k) > 0$ for all k, and (3.12) implies that $\{\epsilon_k\}$ is a nonincreasing monotone sequence. Since $\{\epsilon_k\}$ is nonnegative sequence it follows that it converges. Set $\bar{\epsilon} = \lim_{k\to+\infty}\epsilon_k$. We are going to prove that $\bar{\epsilon} = 0$. First of all, note that from Lemma 3.1.6 the sequence $\{p^k\}$ is Fejér convergent to $S^*(X,\Omega)$ and, due to **A3**, we have $S^*(X,\Omega) \neq \varnothing$. Hence, we conclude that $\{p^k\}$ is bounded. On the other hand, considering that $\{p^k\}$ is bounded, Proposition 1.2.7 implies that $\cup_{k=0}^{\infty} X^{\epsilon_0}(p^k)$ is bounded. Since $\epsilon_k \leq \epsilon_0$, the item $i)$ of Proposition 1.2.5 implies that $X^{\epsilon_k} \subset X^{\epsilon_0}$, for all k. Thus, from (3.4) we conclude that $\{u^k\}$ is also bounded. Definitions of λ_k and y^k in (3.9) implies that y^k belongs the geodesic segment joining p^k to z^k and, using (1.5) and (3.6), we have

$$d(p^k, y^k) \leq d(p^k, z^k) = d\left(P_\Omega(p^k), P_\Omega\left(\exp_{p^k}(-\alpha_k u^k)\right)\right) \leq d(p^k, \exp_{p^k}(-\alpha_k u^k)) = \|\alpha_k u^k\|,$$

for $k = 0, 1, \ldots$. In view of the boundedness of the sequences $\{p^k\}$, $\{u^k\}$ and $\{\alpha_k\}$, we obtain from the last inequalities that $\{y^k\}$ and $\{z^k\}$ are bounded. Considering that $v^k \in X(y^k)$, for all k, we can apply Propositions 1.2.7 and 1.2.2 to conclude that $\{v^k\}$ is bounded. Now, note that the definitions in (3.7), (3.8), (3.9) and (3.10) imply

$$\left\langle v^k, \gamma_k'(\lambda_k)\right\rangle \leq -\frac{\delta_-}{\alpha_k} d^2(p^k, z^k), \qquad k = 0, 1, \ldots.$$

Combining (3.7), (3.8) and (3.9), we conclude that $\gamma'(\lambda_k) = -\lambda_k^{-1}\exp^{-1}_{y^k} p^k$, for $k = 0, 1, \ldots$. Thus, taking into account that $0 < \alpha_k < \alpha_+$, last inequality becomes

$$\left\langle v^k, \exp^{-1}_{y^k} p^k \right\rangle \geq \frac{\lambda_k \delta_-}{\alpha_+} d^2(p^k, z^k), \qquad k = 0, 1, \ldots. \tag{3.19}$$ eq.conv.i

Since $\{p^k\}$, $\{u^k\}$, $\{v^k\}$, $\{z^k\}$, $\{y^k\}$, $\{\alpha_k\}$, and $\{\lambda_k\}$ are bounded, we can assume that they have convergent subsequences $\{p^{k_j}\}$, $\{u^{k_j}\}$, $\{v^{k_j})\}$, $\{z^{k_j}\}$, $\{y^{k_j}\}$, $\{\alpha_{k_j}\}$ and $\{\lambda_{k_j}\}$ with limit $\bar{p}$, $\bar{u}$, $\bar{v}$, $\bar{z}$, $\bar{y}$, $\bar{\alpha}$ and $\bar{\lambda}$, respectively. Note that, (3.11) yields

$$q^{k_j} \in S_{k_j} = \left\{p \in M \ : \ \left\langle v^{k_j}, \exp^{-1}_{y^{k_j}} p\right\rangle \leq 0\right\}, \qquad j = 0, 1, \ldots. \tag{3.20}$$ {?}

Using Lemma 3.1.6 we have $\lim_{j\to\infty} p^{k_j} = \lim_{j\to\infty} q^{k_j} = \bar{p}$. Thereby, latter inequality together with item (iv) of the Lemma 1.0.1 and $\lim_{j\to\infty} y^{k_j} = \bar{y}$ implies

$$\lim_{j\to\infty}\left\langle v^{k_j}, \exp^{-1}_{y^{k_j}} p^{k_j}\right\rangle = \lim_{j\to\infty}\left\langle v^{k_j}, \exp^{-1}_{y^{k_j}} q^{k_j}\right\rangle \leq 0. \tag{3.21}$$ eq.conv.ii

Thus, it follows from (3.19) and (3.21) that

$$\lim_{j\to\infty} \lambda_{k_j} d^2(p^{k_j}, z^{k_j}) = 0. \tag{3.22}$$ Boa_def.iii

Considering that $\lim_{j\to\infty} \lambda_{k_j} = \bar{\lambda}$, we have two possibilities: either $\bar{\lambda} > 0$ or $\bar{\lambda} = 0$. First, let us assume that $\bar{\lambda} > 0$. Since $\lim_{j\to\infty} p^{k_j} = \bar{p}$ and $\lim_{j\to\infty} z^{k_j} = \bar{z}$, it follows from (3.22) that

$$d(\bar{p}, \bar{z}) = \lim_{j\to\infty} d(p^{k_j}, z^{k_j}) = 0, \tag{3.23}$$ {?}

and consequently $\bar{p} = \bar{z}$. Taking into account (3.12), we can apply Lemma 1.0.5 with $\theta_k = \epsilon_k$ and $\rho_k = d^2(p^k, z^k)$ to conclude that $0 = \lim_{k\to+\infty} \epsilon_k = \bar{\epsilon}$. Owing to $u^{k_j} \in X^{\epsilon_{k_j}}(p^{k_j})$, combining Propositions 1.2.6 and 1.2.2 we conclude that $\bar{u} \in X(\bar{p})$. Hence, Lemma 3.1.1 implies that $\bar{p} \in S^*$. Now, let us assume that $\bar{\lambda} = 0$. In this case, using Lemma 1.0.1 and (3.8) we conclude that $\lim_{j\to\infty} y^{k_j, i(k_j)-1} = \bar{p}$. From Proposition 1.1.6 we can take a sequence $\{\xi^j\}$ such that $\xi^j \in X(y^{k_j, i(k_j)-1})$ with $\lim_{j\to\infty} \xi^j = \bar{\xi}$ and, using Proposition 1.1.2, we conclude that $\bar{\xi} \in X(\bar{p})$. On the other hand, (3.7) implies

$$-\left\langle \xi^j, \gamma'_{k_j}(2^{-i(k_j)+1}\beta_{k_j}) \right\rangle < \frac{\delta^-}{\alpha_{k_j}} d^2(p^{k_j}, z^{k_j}), \quad j = 0, 1, \ldots.$$

Considering that $\gamma'_{k_j}(2^{-i(k_j)+1}\beta_{k_j}) = P_{p^{k_j} y^{k_j, i(k_j)-1}} \exp^{-1}_{p^{k_j}} z^{k_j}$, the last inequality becomes

$$-\left\langle \xi^j, P_{p^{k_j} y^{k_j, i(k_j)-1}} \exp^{-1}_{p^{k_j}} z^{k_j} \right\rangle < \frac{\delta^-}{\alpha_{k_j}} d^2(p^{k_j}, z^{k_j}), \quad j = 0, 1, \ldots.$$

Taking limits in the above inequality, as j goes to infinity, and using Lemma 1.0.1 we obtain

$$-\langle \bar{\xi}, \exp^{-1}_{\bar{p}} \bar{z} \rangle \leq \frac{\delta^-}{\bar{\alpha}} d^2(\bar{p}, \bar{z}). \tag{3.24}$$ eq:inqml1

Assume by contradiction that $\bar{\epsilon} > 0$. Theorem 1.2.9 implies that $X^{\bar{\epsilon}}$ is lower semicontinuous. Therefore, due to $\lim_{j\to\infty} p^{k_j} = \bar{p}$ and $\bar{\xi} \in X(\bar{p}) \subset X^{\bar{\epsilon}}(\bar{p})$, there exists a sequence $\{P_{p^{k_j}\bar{p}} w^j\}$ with $w^j \in X^{\bar{\epsilon}}(p^{k_j})$ such that $\lim_{j\to\infty} P_{p^{k_j}\bar{p}} w^j = \bar{\xi}$. Besides, (3.12) implies that $\bar{\epsilon} \leq \epsilon_{k_j}$ and, using item (i) of Proposition 1.2.5, we conclude that $X^{\bar{\epsilon}}(p^{k_j}) \subset X^{\epsilon_{k_j}}(p^{k_j})$, for all j. Henceforth, $w^j \in X^{\epsilon_{k_j}}(p^{k_j})$, for all j, and from (3.5) we have

$$\langle w^j, -\exp^{-1}_{p^{k_j}} z^{k_j} \rangle \geq \frac{\delta^+}{\alpha_{k_j}} d^2(p^{k_j}, z^{k_j}), \quad j = 0, 1, \ldots.$$

Letting j go to infinity in the last inequality and considering Lemma 1.0.1 we obtain

$$-\langle \bar{\xi}, \exp^{-1}_{\bar{p}} \bar{z} \rangle \geq \frac{\delta^+}{\bar{\alpha}} d^2(\bar{p}, \bar{z}).$$

Since $\bar{\alpha} \geq \alpha^- > 0$ and $0 < \delta^- < \delta^+$, combining last inequality with (3.24) we conclude that $\bar{p} = \bar{z}$. Again, taking into account (3.12), we can apply Lemma 1.0.5 with $\theta_k = \epsilon_k$ and $\rho_k = d^2(p^k, z^k)$ to conclude that $0 = \lim_{k \to +\infty} \epsilon_k = \bar{\epsilon}$, which is a contradiction. Due to $u^{k_j} \in X^{\epsilon_{k_j}}(p^{k_j})$, combining Propositions 1.2.6 and 1.2.2 we conclude that $\bar{u} \in X(\bar{p})$. Hence, Lemma 3.1.1 implies that $\bar{p} \in S^*$. The proof is finished. ■

?⟨eq:Conv⟩? **Theorem 3.1.8** *Either the sequence $\{p^k\}$ generated by Algorithm 3.1.3 is finite and ends at iteration k, in which case p^k is ϵ_k-solution of VIP(X, Ω), i.e.,*

$$\sup_{q \in \Omega, v \in X(q)} \left\langle v, \exp_q^{-1} p^k \right\rangle \leq \epsilon_k, \tag{3.25}$$?eq:epssolu?

or it is infinite, in which case it converges to a solution of VIP(X, Ω).

Proof. If Algorithm 3.1.3 stops at the iteration k, then from the stopping criterion we have $p^k = z^k = P_\Omega(\exp_{p^k}(-\alpha_k u^k))$. Since $u^k \in X^{\epsilon_k}(p^k)$ then, using Definition (1.2.1) we obtain

$$-\left\langle u^k, \exp_{p^k}^{-1} q \right\rangle - \left\langle v, \exp_q^{-1} p^k \right\rangle \geq -\epsilon_k, \qquad q \in \Omega, \quad v \in X(q).$$

Since $\alpha_k > 0$ and $p^k = z^k$, the last inequality can be written as

$$\frac{1}{\alpha_k} \left\langle \exp_{z^k}^{-1}[\exp_{p^k}(-\alpha_k u^k)], \exp_{z^k}^{-1} q \right\rangle - \left\langle v, \exp_q^{-1} p^k \right\rangle \geq -\epsilon_k, \qquad q \in \Omega, \quad v \in X(q).$$

In view of (1.4) and considering that $z^k = P_\Omega(\exp_{p^k}(-\alpha_k u^k))$ we conclude from last inequality that

$$\left\langle v, \exp_q^{-1} p^k \right\rangle \leq \epsilon_k, \qquad q \in \Omega, \quad v \in X(q), \tag{3.26}$$ {?}

which implies the desired inequality. Therefore, p^k is an ϵ_k-solution of VIP(X, Ω). Now, if $\{p^k\}$ is infinite, then from Lemma 3.1.6 the sequence $\{p^k\}$ is Féjer convergent to $S^*(X, \Omega)$. Since we are under the assumption **A3**, it follows from Proposition 1.0.3 that $\{p^k\}$ is bounded. Hence, $\{p^k\}$ has a cluster point $\bar{p}$. Using Lemma 3.1.7 we obtain $\bar{p} \in S^*(X, \Omega)$. Therefore, using again Proposition 1.0.3 we conclude that $\{p^k\}$ converges to $\bar{p} \in S^*(X, \Omega)$ and the theorem is proved. ■

3.2 Remarks

The concept of approximate solutions of VIP(X, Ω) is related to an important function, namely, the *gap function* $h : \Omega \to \mathbb{R} \cup \{+\infty\}$ defined by

$$h(p) := \sup_{q \in \Omega, v \in X(q)} \left\langle v, \exp_q^{-1} p \right\rangle. \tag{3.27}$$ gap

The relation between the function h and the solutions of VIP(X, Ω) is given in the following lemma, which is a Riemannian version of [15, Lemma 4].

Proposition 3.2.1 *Let h be the function defined in* (3.27)*. Then, there holds* $h^{-1}(0) = S^*(X,\Omega)$.

Proof. We will see first that a zero of h is a solution of VIP(X,Ω), i.e, $h^{-1}(0) \subset S^*(X,\Omega)$. Let $p \in h^{-1}(0)$. Thus, $h(p) = 0$ and the definition of h in (3.27) implies

$$\left\langle v, \exp_q^{-1} p \right\rangle \le 0, \qquad q \in \Omega, \quad v \in X(q).$$

On the other hand, from the definition of normal cone N_Ω in (1.6), we have

$$\left\langle w, \exp_q^{-1} p \right\rangle \le 0, \qquad q \in \Omega, \quad w \in N_\Omega(q),$$

Combining this two last inequalities it is easy to conclude that

$$\left\langle 0 - (v + w), \exp_q^{-1} p \right\rangle \ge 0, \qquad q \in \Omega, \quad v \in X(q), \quad w \in N_\Omega(q).$$

Due to Lemma 1.1.5, the vector field $X + N_\Omega$ is maximal monotone. Then, the maximality property together with latter inequality yields $0 \in X(p) + N_\Omega(p)$, i.e., $p \in S^*(X,\Omega)$.

Now, we are going to show that the solutions of VIP(X,Ω) are zeros of h, i.e, $S^*(X,\Omega) \subset h^{-1}(0)$. Suppose that $p \in S^*(X,\Omega)$. Then, there exists $u \in X(p)$ such that

$$\langle u, \exp_p^{-1} q \rangle \ge 0, \qquad q \in \Omega.$$

Using the last inequality and the monotonicity of the vector field X we obtain

$$\langle v, \exp_q^{-1} p \rangle \le 0, \qquad q \in \Omega, \quad v \in X(q).$$

Therefore, definition of h in (3.27) implies $h(p) \le 0$ and, considering that $h(p) \ge 0$, we conclude that $h(p) = 0$, which ends the proof. ■

In linear spaces, the gap function is convex. Thus, it is quite common to use this connection with the problem of minimization the gap function to explore variational inequalities. However, at least to our knowledge, the convexity of the gap function in Hadamard manifolds is still a doubtful question, which greatly compromises the analysis of the gap function in this context.

Chapter 4

An Existence Result for the Generalized Vector Equilibrium Problem on Hadamard Manifolds

⟨chapter4⟩

The generalized vector equilibrium problem (GVEP) has been widely studied and continues to be an active topic for research. One of the primary reasons for this is that multiple problems can be formulated as generalized vector equilibrium problems, such as optimization, vector optimization, Nash equilibria, complementarity, fixed point, and variational inequality problems. Extensive developments of these problems can be found in Fu [31], Fu and Wan [32], Konnov and Yao [39], Ansari et al. [3], Farajzadeh et al. [28], and the references therein. An important question concerns the conditions under which a solution to the GVEP exists. In a linear setting, multiple authors have provided results that answer this question, such as Ansari and Yao [4], Fu [31], Fu and Wan [32], Konnov and Yao [39], Ansari et al. [3], Farajzadeh et al. [28]. Moreover, it should be noted that Ky Fan studied inequalities in [26], which prompted present equilibrium theory.

Colao et al. [19] and Zhou and Huang [62] were the first authors to examine the existence of solutions for equilibrium problems in the Riemannian context by generalizing the Knaster-Kuratowski-Mazurkiewicz (KKM) Lemma to Hadamard manifolds. Applying the KKM Lemma in a Riemannian setting allowed Zhou and Huang [44] to confirm solution existence for vector optimization problems and vector variational inequalities in this context. Similarly, Li and Huang [61] presented results concerning solution existence for a special class of GVEP. In this paper, we apply the KKM Lemma in a Riemannian setting in order to prove solution existence for GVEP. To the best of our knowledge, our contribution is unprecedented. However, it should be noted that the results of this paper include the results presented in [19, 44] and are not included in [61].

4.1 An Existence Result for the Generalized Vector Equilibrium Problem

In this Section, we present a sufficient condition for the existence of a solution to the generalized vector equilibrium problem on Hadamard manifolds. We should note that this material is motivated by the results found in [4]. Henceforth, we let $\Omega \subseteq M$ denote a nonempty, closed and convex set, $\mathbb{Y}$ denote a metric vector space and $C : \Omega \rightrightarrows \mathbb{Y}$ denote a set-valued mapping such that

$$C(x) \ \text{ is a closed and convex cone,} \qquad \operatorname{int} C(x) \neq \varnothing, \qquad \forall\, x \in \Omega. \tag{4.1}$$?eq:cx?

Suppose $x \in \Omega$. A set-valued mapping $F : \Omega \times \Omega \rightrightarrows \mathbb{Y}$ is called $C(x)$ - *quasiconvex-like* iff for any geodesic segment $\gamma : [0,1] \to \Omega$, either $F(x, \gamma(t)) \subseteq F(x, \gamma(0)) - C(x)$ or $F(x, \gamma(t)) \subseteq F(x, \gamma(1)) - C(x)$, for all $t \in [0,1]$.

⟨subEx1⟩ **Example 4.1.1** Let $(\mathbb{H}^2, g_{ij})$ be the 2-dimensional hyperbolic space, as defined in Example 2.1.2. In addition, assume that $F : \mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{R}$ is the bifunction given by

$$F\left((x_1, x_2), (y_1, y_2)\right) = \left| y_1^2 + y_2^2 - x_1^2 - x_2^2 \right|.$$

Since, for every $c \in \mathbb{R}$, the sub-level set

$$L_{\psi,\Omega}(c) = \left\{ (y_1, y_2) \in \mathbb{R}^2 \ : \ -c + x_1^2 + x_2^2 \leq y_1^2 + y_2^2 \leq c + x_1^2 + x_2^2, \ y_2 > 0 \right\},$$

is convex in $\mathbb{H}^2$, where $\psi(y_1, y_2) = F((x_1, x_2), (y_1, y_2))$ and $(x_1, x_2) \in \Omega$ is a fixed point, we can conclude that F is $C(x)$ - quasiconvex-like. It should be noted that F is not $C(x)$ - quasiconvex-like in the Euclidean setting.

Given a set-valued mapping $F : \Omega \times \Omega \rightrightarrows \mathbb{Y}$, the *generalized vector equilibrium problem* (GVEP) in the Riemannian context consists in

$$\text{Find } x^* \in \Omega : \quad F(x^*, y) \nsubseteq -\operatorname{int} C(x^*), \qquad \forall\, y \in \Omega. \tag{4.2}$$ eq:p

Remark 4.1.2 Let $M = \mathbb{R}^n$, $\mathbb{Y} = \mathbb{R}^m$ and $\operatorname{int} C(x) = K$ for all $x \in \mathbb{R}^n$, where $K \subset \mathbb{R}^m$ is a closed pointed and convex cone such that $\operatorname{int} K \neq \varnothing$. Given a function $f : \mathbb{R}^n \to \mathbb{R}^m$, if $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ is defined by $F(x, y) = f(y) - f(x)$, then we can tranform the GVEP in (4.2) into the classic vector optimization problem $\min_K f(x)$; see [33].

Remark 4.1.3 Although variational inequality theory provides us with a toll for formulating multiple equilibrium problems, Iusem and Sosa [37, Proposition 2.6] demonstrated that the generalization given by equilibrium problem (EP) formulation with respect to variational inequality (VI) is genuine, meaning there are EP formulations that do not fit the format of a VI. When compared with VIs, EP formulations may also guarantee genuineness by considering the important class of quasiconvex optimization problems, which appear, for instance, in many micro-economical models that are devoted to maximizing utility. Indeed, the absence of convexity allows us to obtain situations in which this important class of problems cannot be considered to be a VI because its possible representation given this format produces a problem whose solution set contains points that do not necessarily belong to the solution set of the original optimization problem. For example, let $\Omega \subseteq M$ be a nonempty, closed and convex set, and $f : M \to \mathbb{R}$ be a differentiable and $(\mathbb{R}_+)$-quasiconvex-like function. Consider the following optimizations problem:

$$\text{Find } x^* \in \Omega : \quad f(y) - f(x^*) \notin -\operatorname{int}\mathbb{R}_+, \qquad \forall\, y \in \Omega. \tag{4.3}$$ eq:mp

Note that, if $F : \Omega \times \Omega \to \mathbb{R}$ is the bifunction given by $F(x, y) = f(y) - f(x^*)$, then the optimization problem in (4.4) is equivalent to the following equilibrium problem:

$$\text{Find } x^* \in \Omega : \quad F(x^*, y) \notin -\operatorname{int}\mathbb{R}_+, \qquad \forall\, y \in \Omega. \tag{4.4}$$ eq:mp

On the other hand, in the absence of convexity, the optimization problem in (4.4) is not equivalent to the associated variational inequality,

$$\text{Find } x^* \in \Omega : \quad \langle \nabla f(x^*), y - x^* \rangle \notin -\operatorname{int}\mathbb{R}_+, \qquad \forall\, y \in \Omega,$$

because, for instance, point $x^* \in \Omega$, in which $\nabla f(x^*) = 0$, is a solution to this variational inequality, but it cannot be a solution to the equilibrium problem in (4.4).

The following result is closely related to [4, Theorem 2.1] and establishes an existence result of solution for GVEP as an application of Lemma 1.0.2.

⟨th:main⟩ **Theorem 4.1.4** *Let $F : \Omega \times \Omega \rightrightarrows \mathbb{Y}$ be a set-valued mapping such that, for each $x, y \in \Omega$, we have:*

h1. $F(x, x) \not\subset -\operatorname{int} C(x)$;

h2. $F(\cdot, y)$ *is upper semicontinuous;*

h3. F *is* $C(x)$*-quasiconvex-like;*

h4. *there exist* $D \subset \Omega$ *compact and* $y_0 \in \Omega$ *such that* $x \in \Omega \backslash D \Rightarrow F(x, y_0) \subset -\operatorname{int} C(x)$.

Then, the solution set, S^*, of the GVEP defined in (4.2) is a nonempty compact set.

Remark 4.1.5 In particular, when $M = \mathbb{R}^n$, problem (4.2) retrieves a particular instance of the generalized vector equilibrium problem studied in [4]. In the case where $C(x) = \mathbb{R}_+$, for each $x \in \Omega$ fixed, $\mathbb{Y} = \mathbb{R}$ and F is single-valued map from $\Omega \times \Omega$ to $\mathbb{R}$, then problem (4.2) reduces to the equilibrium problem on Hadamard manifold that was studied in [19]. Let us consider the following vector optimization problem on Hadamard manifolds:

$$\min_{\mathbb{R}^m_+} f(x), \qquad \text{such that} \quad x \in \Omega, \tag{4.5}$$ eq:vopt

in which $f : M \to \mathbb{R}^m$ is a vector function and $\min_{\mathbb{R}^m_+}$ represents the weak minimum. In the main result of [44], namely, Theorem 3.2, the existence of a solution to (4.5) was achieved by demonstrating the equivalence of this and the variational inequality on Hadamard manifolds (studied by Németh in [47]):

$$\text{Find } x^* \in \Omega : \quad \langle A(x^*), \exp^{-1}_{x^*} y \rangle \notin -\mathbb{R}^m_{++}, \qquad \forall\, y \in \Omega, \tag{4.6}$$?prob:VVI?

in the particular case where f is a differentiable and convex vector function and A is the Riemannian Jacobian of f. When we consider that $x^* \in \Omega$ is a weak minimum of (4.5), i.e., $f(x) - f(x^*) \notin -\mathbb{R}^m_{++}$, for all $x \in \Omega$, then Theorem 4.1.4 increases the applicability of [44, Theorem 3.2] to genuine Hadamard manifolds and quasi-convex non-differentiable vector functions.

⟨subEx⟩

Example 4.1.6 Let $(\mathbb{H}^2, g_{ij})$ be the 2-dimensional hyperbolic space, as defined in Example 4.1.1. The bifunction $F : \mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{R}$, which is given by $F((x_1, x_2), (y_1, y_2)) = \ln^2(y_1^2 + y_2^2) - \ln^2(x_1^2 + x_2^2)$, satisfies all the assumptions in Theorem 4.1.4 if $\Omega = \{x = (x_1, x_2) \in \mathbb{H}^2 \ : \ x_2 \geq 1/2\}$, $C(x) \equiv \mathbb{R}_+$, $y_0 = (0, 1)$, and

$$D = \left\{(x_1, x_2) \in \mathbb{H}^2 : x_1^2 + x_2^2 \leq 1,\ x_2 \geq 1/2\right\}.$$

Indeed, it is clear that $F((x_1, x_2), (x_1, x_2)) = 0$ for all $(x_1, x_2) \in \Omega$, which implies that F satisfies **h1**. In addition, for fixed $(y_1, y_2) \in \Omega$, we know that $\varphi(x_1, x_2) = F((x_1, x_2), (y_1, y_2))$ is continuous, and F consequently satisfies **h2**. Moreover, for all $c \in \mathbb{R}$, the sub-level set,

$$L_{\psi,\Omega}(c) = \left\{(y_1, y_2) \in \mathbb{R}^2 \ : \ e^{-\sqrt{d}} \leq y_1^2 + y_2^2 \leq e^{\sqrt{d}},\ y_2 > 0\right\}, \qquad d = c + \ln^2\left(x_1^2 + x_2^2\right),$$

is convex in $\mathbb{H}^2$, where $\psi(y_1, y_2) = F((x_1, x_2), (y_1, y_2))$, and $(x_1, x_2) \in \Omega$ is a fixed point. Hence, F satisfies **h3**. Finally, because we have $F((x_1, x_2), (0, 1)) < 0$ for all $x \in \Omega \backslash D$, then we know that F satisfies **h4**. Moreover, according to Theorem 4.1.4, we can conclude that $S^* = \{(x_1, x_2) \in \mathbb{H}^2 : x_1^2 + x_2^2 = 1,\ x_2 \geq 1/2\}$, and the set is compact.

Remark 4.1.7 One reason for the successful extension, to the Riemannian setting, is the possibility to transform nonconvex or quasi-convex problems in linear context into convex or quasi-convex problems by introducing a suitable metric; see Rapcsák [50]. For instance, in Example 4.1.6, for a fixed point $(x_1, x_2) \in \Omega$, the function $\psi(y_1, y_2) = \ln^2(y_1^2 + y_2^2) - \ln^2(x_1^2 + x_2^2)$ is not usual quasi-convex in $\{(y_1, y_2) \in \mathbb{R}^2 : y_2 > 0\}$, because its sub-level $L_{\psi,\Omega}(0) = \{(y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 = 1, \ y_2 > 0\}$ is not convex. Therefore, [4, Theorem 2.1] cannot be applied to the GVEP. However, we can apply Theorem 4.1.4.

Henceforth, we assume that assumptions made in Theorem 4.1.4 hold. In order to prove this theorem, we must establish some preliminary concepts. First, we define the set-valued mapping, $P : \Omega \rightrightarrows \Omega$, by

$$P(x) := \{y \in \Omega \ : \ F(x, y) \subset -\operatorname{int} C(x)\}. \tag{4.7}$$ eq:set

⟨l:at1⟩ **Lemma 4.1.8** *If $S^* = \varnothing$, then for each $x, y \in \Omega$, the set-valued mapping P satisfies the following conditions:*

(i) set $P(x)$ is nonempty and convex;

(ii) $P^{-1}(y)$ is an open set, and $\bigcup_{y\in\Omega} P^{-1}(y) = \Omega$;

(iii) there exists $y_0 \in \Omega$ such that $P^{-1}(y_0)^c$ is compact.

Proof. Because the solution set $S^* = \varnothing$, the definition in (4.7) lets us to conclude that $P(x) \neq \varnothing$, for all $x \in \Omega$, which proves the first statement in (i). Assume $x \in \Omega$. To prove $P(x)$ is convex, we consider $y_1, y_2 \in P(x)$ and a geodesic $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = y_1$ and $\gamma(1) = y_2$. Applying assumption **h3** we find

$$F(x, \gamma(t)) \subseteq F(x, y_1) - C(x) \quad \text{or} \quad F(x, \gamma(t)) \subseteq F(x, y_2) - C(x). \tag{4.8}$$ eq:itws

As $y_1, y_2 \in P(x)$, the definition of $P(x)$ in (4.7) implies that $F(x, y_1) \subset -\operatorname{int} C(x)$ and $F(x, y_2) \subset -\operatorname{int} C(x)$. Therefore, given $-\operatorname{int} C(x) - C(x) \subset -\operatorname{int} C(x)$, which is obtained using Proposition 1.3 and Proposition 1.4 of [45], it follows from (4.8) that $F(x, \gamma(t)) \subset -\operatorname{int} C(x)$, and this concludes the proof of (i).

In order to prove (ii), we must first note that the definition in (1.8) provides

$$P^{-1}(y) = \{x \in \Omega \ : \ y \in P(x)\} = \{x \in \Omega \ : \ F(x, y) \subset -\operatorname{int} C(x)\}, \tag{4.9}$$ eq:pm1

where the second equality follows from the definition of the set, $P(x)$, in (4.7). Given $x_0 \in P^{-1}(y)$, the second equality in (4.9), and the fact that $-\operatorname{int} C(x)$ is an open set, if we apply **h2**, then we know there exists an open set, $V_{x_0} \subset \Omega$, such that $F(x, y) \subset -\operatorname{int} C(x)$, for all $x \in V_{x_0}$. Hence, $P^{-1}(y)$ is open, which proves the first statement in (ii). The definition in (4.9) implies that $P^{-1}(y) \subseteq \Omega$ for all $y \in \Omega$. In order to complete the proof of (ii), it is sufficient to prove that $\Omega \subseteq \bigcup_{y\in\Omega} P^{-1}(y)$. Therefore, suppose $x \in \Omega$. Item (i) ensures that $P(x) \neq \varnothing$, which implies that there exists $y \in P(x)$. Thus, $x \in P^{-1}(y)$ for some $y \in \Omega$, which concludes the proof of item (ii).

To prove (iii), we note that **h4** and (4.9) imply that $P^{-1}(y_0)^c = \{x \in \Omega \ : \ F(x, y_0) \not\subset -\operatorname{int} C(x)\} \subset D$, for some $y_0 \in \Omega$, and $D \subset \Omega$ is a compact set. Given item (i), we know $P^{-1}(y_0)$ is an open set. Furthermore, because D is compact, we can conclude from the last inclusion that $P^{-1}(y_0)^c$ is a compact set, and this completes the proof of the Lemma. ■

Now we are ready to prove our main result in this section: Theorem 4.1.4.

Proof. In order to create a contradiction, let us suppose that $S^* = \varnothing$. Also, assume $G : \Omega \rightrightarrows \Omega$ is the set-valued mapping defined by

$$G(y) := P^{-1}(y)^c. \tag{4.10}$$ eq:faux

Further define set $D := \bigcap_{y\in\Omega} G(y)$. Therefore, we have two possibilities for set D: $D \neq \varnothing$ or $D = \varnothing$. If $D \neq \varnothing$, i.e., $\bigcap_{y\in\Omega} P^{-1}(y)^c \neq \varnothing$, then we have $\bigcup_{y\in\Omega} P^{-1}(y) \neq \Omega$, which contradicts (ii) in Lemma 4.1.8. Hence, we can conclude that $D = \varnothing$, i. e., $\bigcap_{y\in\Omega} G(y) = \varnothing$.

Thus, given our assumption that $S^* = \varnothing$, combining the definition in (4.10) and statements (ii) and (iii) in Lemma 4.1.8, we conclude that, for each $y \in \Omega$, set $G(y)$ is closed, and there exists $y_0 \in \Omega$ such that $G(y_0)$ is a compact set. Hence, because $\bigcap_{y\in\Omega} G(y) = \varnothing$, Lemma 1.0.2 implies that there exist $y_1, ..., y_m \in \Omega$ such that $\operatorname{conv}\{y_1, ..., y_m\} \not\subset \bigcup_{i=1}^{m} G(y_i)$. Therefore, there also exists $x \in \operatorname{conv}\{y_1, ..., y_m\}$ such that $x \notin G(y_i) = P^{-1}(y_i)^c$ for all $i = 1, \ldots m$. Equivalently, there exists $x \in \operatorname{conv}\{y_1, ..., y_m\}$ such that $x \in P^{-1}(y_i)$ for all $i = 1, \ldots m$. Hence, we conclude that

$$\exists\ y_1, ..., y_m \in \Omega, \quad \exists\ x \in \operatorname{conv}\{y_1, ..., y_m\}; \quad y_i \in P(x), \quad \forall\ i = 1, \ldots m. \tag{4.11}$$ eq:rmth

Considering $S^* = \varnothing$, items (i) in Lemma 4.1.8 implies that $P(x)$ is convex. When combined with the relations in (4.11), this implies that there exists $x \in \Omega$ such that $x \in P(x)$. These inclusions and the definition in (4.7) imply that there exists $x \in \Omega$ such that $F(x, x) \subset -\operatorname{int} C(x)$. This contradicts assumption **h1** in Theorem 4.1.4. Therefore, solution set $S^* \neq \varnothing$, and this concludes the proof of Theorem 4.1.4. ■

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