Q-QUADRATIC CONVERGENCE ON NEWTON'S METHOD
FROM DATA AT ONE POINT

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Abstract: Smale's Theorem on Newton's Method for analytic systems provides existence of a solution and $R$-quadratic convergence of the method from data at one point. In this paper, we prove that Newton Method under Smale's hypothesis is $Q$-quadratic convergent and as a consequence, we deduce an error estimate.

Key Words: Newton's Method, Smale's hypothesis, $q$-quadratic convergence

1. Introduction

By "solving" $F(x) = 0$ we shall understand to get an "approximated solution". i.e., to get a point $x_0$ such that Newton Method for solving $F(x) = 0$, with starting point $x_0$ generates a sequence that converges to a solution. In particular, an "approximated solution" implies the existence of a solution. The most important question is to decide whether a given point is an approximated solution. Smale in [4] has given conditions under which a point is an approximated solution using only the information available at the starting point.

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The aim of this paper is to prove the $q$-quadratic convergence of Newton Method under Smale's conditions, which is a new result, once up to now only $r$-quadratic convergence was proved. As a consequence, we deduce an error estimate.

2. Auxiliary Results

Let $E$ and $F$ be Banach spaces and $f: D_r(x_0) \rightarrow F$ be an analytic map, where $x_0 \in E$ and $D_r(x_0) = \{ x \in E : \| x - x_0 \| \leq r \}$. The derivative of $f$ at $x \in D_r(x_0)$ will be denoted by $Df(x)$ (and the higher order derivatives by $D^k f(x)$). Newton Method for solving

\[ f(x) = 0 \]

generates the sequence $\{ x_n \}$ by the iterative process

\[ x_n = x_{n-1} - Df(x_{n-1})^{-1} f(x_{n-1}) \]

provided that, for all $n \geq 1$, $Df(x_k)^{-1}$ exists. In [4], Smale studied the Newton Method in this context and deduced consequences from data at a single point, but only $R$-quadratic convergence, see Ortega et al [1], is obtained. As in Shub at al [3], we use the same notation to obtain $Q$-quadratic convergence, see Ortega et al [1], of Newton Method for this context.

For every point $x \in D_r(x_0)$ define

\[ \beta(f, x) = \| Df(x)^{-1} f(x) \|, \quad \gamma(f, x) = \sup_{k \geq 1} \left\| \frac{Df(x)^{-1} D^k f(x)}{k!} \right\|^{\frac{1}{k+1}} \]

and if $Df(x)^{-1}$ does not exist, define $\beta(f, x) = \infty$ and $\gamma(f, x) = \infty$. Now define, $\alpha(f, x) = \beta(f, x) \gamma(f, x)$.

The following expressions play an important role in the next results:

\[ \tau(\alpha) = \frac{(1 + \alpha) - \sqrt{(1 + \alpha)^2 - 8\alpha}}{4}, \quad \text{for } 0 \leq \alpha \leq 3 - 2\sqrt{2}, \]

\[ \alpha_0 = \frac{1}{4}(13 - 3\sqrt{17}). \]

A point $x_0$ is called an approximated zero of $f$, if the sequence $\{ x_n \}$, generated by (2), is well defined and satisfies:

\[ \| x_n - x_{n-1} \| \leq \left( \frac{\sqrt{2}}{2} \right)^{\alpha - 1} \| x_1 - x_0 \| \]

for all $n \geq 1$. The next result gives us conditions under which a point $x_0$ is an approximated zero, for the proof see Shub at al [3].
Theorem 1. Let \( f : D_r(x_0) \rightarrow \mathbb{R} \) be analytic map, \( \beta = \beta(f, x_0), \gamma = \gamma(f, x_0), \alpha = \beta\gamma \) and \( r \geq \frac{\tau(\alpha)}{\gamma} \). Then if \( \alpha \leq \alpha_0 \), the Newton iterates \( x_1, x_2, \ldots \) are defined well, converge to \( \zeta \in D_r(x_0) \) with \( f(\zeta) = 0 \) and for all \( n \geq 1 \)

\[
||x_n - x_{n-1}|| \leq \left( \frac{1}{2} \right)^{2^{n-1}-1} ||x_1 - x_0||.
\]

(7)

Moreover, \( ||\zeta - x_0|| \leq \frac{\tau(\alpha)}{\gamma} \), and \( ||\zeta - x_1|| \leq \frac{\tau(\alpha) - \alpha}{\gamma} \).

Theorem 1 implies that the sequence \( \{x_k\} \) satisfies

\[
||x_n - \zeta|| \leq \left( \frac{1}{2} \right)^n ||x_1 - x_0||K,
\]

(8)

where \( K = \sum_{n=1}^{\infty} \left( \frac{1}{2} \right)^{2^{n-1}-1} \), see Smale [4], and this inequality signifies that the sequence \( \{x_k\} \) has convergence R-quadratic.

Now, for each \( \beta, \gamma > 0 \), define

\[
h_{\beta, \gamma}(t) = \beta - t + \frac{\gamma t^2}{1 - \gamma t}.
\]

(9)

Let \( \alpha = \beta\gamma \) satisfy \( (\alpha + 1) - 8\alpha > 0 \) or equivalently \( 0 < \alpha < 3 - 2\sqrt{2} \). Then \( h_{\beta, \gamma}(t) = 0 \) has two distinct real positive roots, the smaller root is

\[
\frac{\tau(\alpha)}{\gamma} = \frac{(1 + \alpha) - \sqrt{(1 + \alpha)^2 - 8\alpha}}{4\gamma}.
\]

(10)

Moreover \( d^2h_{\beta, \gamma}/dt^2(t) > 0 \) as long as \( 0 < t < \frac{1}{\gamma} \) which implies that \( h_{\beta, \gamma} \) is convex in this interval. Thus Newton Method, to solving \( h_{\beta, \gamma}(t) = 0 \), starting at \( t_0 = 0 \) generates the monotone sequence \( \{t_n\} \) which converges to \( \frac{\tau(\alpha)}{\gamma} \).

Theorem 2. (Domination Theorem) Let \( f : D_r(x_0) \rightarrow \mathbb{R} \) be analytic map, \( \beta = \beta(f, x_0), \gamma = \gamma(f, x_0), \alpha = \beta\gamma \) and suppose \( r \geq \frac{\tau(\alpha)}{\gamma} \) and \( \alpha \leq \alpha_0 \). These values of \( \beta, \alpha \) define \( h_{\beta, \gamma} \) and the sequence \( \{t_n\} \). Then

\[
||x_n - x_{n-1}|| \leq t_n - t_{n-1}, \quad n = 1, 2, \ldots,
\]

(11)

where \( \{x_n\} \) is the Newton sequence of \( f \) starting at \( x_0 \).

It follows from (11) that

\[
||x_n - x_0|| \leq t_n, \quad n = 1, 2, \ldots
\]

(12)

and this implies that \( \{x_n\} \subset D_{\frac{\tau(\alpha)}{\gamma}}(x_0) \).
Let
\[ \psi(u) = 2u^2 - 4u + 1, \quad 0 \leq u \leq 1 - \frac{\sqrt{2}}{2}, \] (13)
so that \( 0 \leq \psi(u) \leq 1. \)

**Lemma 1.** Let \( f : D_r(x_0) \rightarrow \mathbb{F} \) be analytic map and let \( \gamma = \gamma(f, x_0) \). If \( x \in D_r(x_0) \) with \( \psi(u) > 0 \), where \( u = ||x - x_0||/\gamma \), then
(1) \( Df(x) \) is invertible;
(2) \( ||Df(x)^{-1}Df(x_0)|| \leq \frac{(1-u)^2}{\psi(u)}. \)

**Proof.** See Shub at al [3], Lemma 3, p. 476. \( \square \)

We observe that
\[ \frac{(1-u)^2}{\psi(u)} = -\frac{1}{h_{r,\gamma}^{\alpha}(u)}. \] (14)

Since \( h_{r,\gamma}^{\alpha} \) is monotone, then from (14) it follows that, for all \( \alpha \leq \alpha_0 \), and \( x \in D_{\frac{\tau(\alpha)}{r}}(x_0) = \{ x \in E : ||x - x_0|| \leq \frac{\tau(\alpha)}{r} \}, \)
\[ \frac{(1-||x - x_0||/\gamma)^2}{\psi(||x - x_0||/\gamma)} \leq \frac{(1-\tau(\alpha))^2}{\psi(\tau(\alpha))}. \] (15)

**3. Q-Quadratic Convergence**

This is the main section. Here we prove that, under Smale's Conditions, the sequence generated by Newton Method \( Q \)-quadratically converges and as a consequence, we deduce an error estimate.

**Lemma 2. (Lemma of Calculus)** Let \( f : D_r(x_0) \rightarrow \mathbb{F} \) be continuous, differentiable in the interior \( D_r^*(x_0) \) of \( D_r(x_0) \) and \( Df(x_0) \) be non-singular. Suppose that, for all \( x, x' \in D_r(x_0) \)
\[ ||Df(x_0)^{-1}(Df(x) - Df(x'))|| \leq L ||x' - x||. \]

If \( x \in D_r^*(x_0), v \in \mathbb{E}, t \in R \) and \( x + tv \in D_r(x_0) \), then
\[ f(x + tv) = f(x) + tDf(x) v + R(t) \quad \text{with} \quad ||Df(x_0)^{-1}R(t)|| \leq L t^2 ||v||^2. \]

**Proof.** Follows from "Fundamental Theorem of Calculus". \( \square \)
Lemma 3. Let $f : D_r(x_0) \to F$ be analytic map, $\beta = \beta(f, x_0), \gamma = \gamma(f, x_0), \alpha = \beta \gamma$ and $r > r(\alpha)$. If $\alpha \leq \alpha_0$ then, for all $x, x' \in D_{r(\alpha)}(x_0)$

$$
||Df(x_0)^{-1}(Df(x') - Df(x))|| \leq \frac{2\gamma}{(1 - \tau(\alpha))^3} ||x' - x||.
$$

(16)

Proof. Let $w \in D_{r(\alpha)}(x_0)$. Note that

$$
Df(x_0)^{-1}D^2f(w) = \sum_{0}^{\infty} \frac{1}{k!} Df(x_0)^{-1}D^{k+2}f(x_0)(w - x_0)^k.
$$

(17)

Since $\alpha \leq \alpha_0$ we have that $\gamma ||w - x_0|| \leq \tau(\alpha) < 1$, thus from (17) it follows that

$$
||Df(x_0)^{-1}D^2f(w)|| \leq \gamma \sum_{0}^{\infty} \frac{(k + 2)(k + 1)(\gamma ||w - x_0||)^k}{(1 - \gamma ||w - x_0||)^3}
$$

(18)

But since

$$
||Df(x_0)^{-1}(Df(x) - Df(x'))|| \leq \sup_{w \in D_r(x_0)} ||Df(x_0)^{-1}D^2f(w)|| ||x' - x||,
$$

it follows from (18) the statement of the lemma.

Theorem 3. Let $f : D_r(x_0) \to F$ be analytic map, $\beta = \beta(f, x_0), \gamma = \gamma(f, x_0), \alpha = \beta \gamma$ and $r > r(\alpha)$. Then if $\alpha \leq \alpha_0$, the Newton iterates $x_1, x_2, \ldots$ are defined well, converge to $\zeta \in D_r(x_0)$ with $f(\zeta) = 0$ and there exists a constant $M = M(x_0)$ such that

$$
||x_{n+1} - \zeta|| \leq M||x_n - \zeta||^2
$$

for all $n \geq 1$.

Proof. Since $\alpha \leq \alpha_0$ from Theorem 1 it follows that $x_0$ is an approximated zero of $f$, then the sequence $\{x_n\}$ converges to $\zeta$, where $f(\zeta) = 0$ and from equation (12) the sequence $\{x_n\} \subset D_{r(\alpha)}(x_0)$. Furthermore, from Lemma 1 and (15), it follows that

$$
||Df(x_0)^{-1}Df(x_0)|| \leq \frac{(1 - \tau(\alpha))^2}{\psi(\tau(\alpha))}.
$$

(19)
Now from Lemma 2 and Lemma 3 it follows that
\[ ||DF(x_0)^{-1}R_n|| \leq \frac{2\gamma}{2(1 - \tau(\alpha))^3} ||x_n - \zeta||^2, \] (20)
where
\[ f(\zeta) = f(x_n) + Df(x_n)(\zeta - x_n) + R_n. \] (21)
Thus the inequalities (19), (20) and \( f(\zeta) = 0 \) imply that
\[ ||x_{n+1} - \zeta|| \leq ||Df(x_n)^{-1}Df(x_0)|| \cdot ||Df(x_0)^{-1}R_n|| \leq M||x_n - \zeta||^2, \] (22)
where
\[ M = \frac{(1 - \tau(\alpha))^2}{\psi(\tau(\alpha))} \cdot \frac{2\gamma}{2(1 - \tau(\alpha))^3} = \frac{\gamma}{\psi(\tau(\alpha))(1 - \tau(\alpha))}. \]

**Theorem 4.** Let \( \{x_n\} \) be a sequence in Banach space \( E \), convergent to \( \zeta \) such that
\[ ||x_{n+1} - \zeta|| \leq a||x_n - \zeta||^2 \] (23)
for all \( n \) and a positive constant \( a \). If \( \mu < 1/4 \), \( a||x_{n+1} - x_n|| < \mu \) and
\[ a||\zeta - x_n|| \leq \frac{2}{1 + \sqrt{1 + 4\mu}}, \]
then
\[ \frac{2}{1 + \sqrt{1 + 4\mu}} \leq ||\zeta - x_n|| \leq \frac{2}{1 + \sqrt{1 + 4\mu}}. \]

**Proof.** See Ostrowski [2], pp. 372, 373. \( \Box \)

From Theorem 3 and Theorem 4 we obtain the following theorem.

**Theorem 5.** Let \( f: D_r(x_0) \rightarrow E \) be analytic map, \( \beta = \beta(f, x_0), \gamma = \gamma(f, x_0), \ \alpha = \beta \gamma \) and \( r \geq \frac{\tau(\alpha)}{\gamma} \). Then if \( \alpha \leq \alpha_0 \), the Newton iterates \( x_1, x_2, \ldots \) are defined well, converge to \( \zeta \in D_r(x_0) \) with \( f(\zeta) = 0 \) and there exists a constant \( \mu \leq .115146 \) such that
\[ \frac{2}{1 + \sqrt{1 + 4\mu}} \leq ||\zeta - x_{n+1}|| \leq \frac{2}{1 + \sqrt{1 + 4\mu}}, \] (24)
for all \( n \geq 2 \).

**Proof.** From Theorem 3 follows that \( \{x_n\} \) satisfies (23) with \( a = M \).
Define \( \mu = \frac{\alpha}{8\psi(\tau(\alpha))(1 - \tau(\alpha))} \leq .115146 < 1/4 \), where the functions \( \tau \) and \( \psi \) were defined respectively in (10) and (13). Now by (7)
\[ M||x_{n+1} - x_n|| \leq \frac{2M\beta}{2\psi} \leq \frac{\alpha}{8\psi(\tau(\alpha))(1 - \tau(\alpha))} \leq \mu. \]
For $n > 2$, and by (8) $\|z - x_n\| \leq \frac{K^2}{2W+1}$, where $K \leq \frac{7}{4}$, we have

$$M\|x_0 - x_1\| \leq \frac{7\alpha}{16\psi(\tau(\alpha))(1 - \tau(\alpha))} \leq \frac{7\mu}{2} \leq \frac{1 + \sqrt{1 - 4\mu}}{2}.$$ 

Thus the statement of the theorem follows from Theorem 4.

References


